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Random Infinite-Volume Gibbs States for the Curie-Weiss Random Field Ising Model

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Abstract. An approach to the definition of infinite-volume Gibbs states for the (quenched) random-field Ising model is considered in the case of a Curie-Weiss ferromagnet. It turns out that these states are random quasi-free measures. They are random convex linear combinations of the free product-measures "shifted" by the corresponding effective mean fields. The conditional self-averaging property of the magnetization related to this randomness is also discussed.

KEY WORDS: Random Field, Ising Model, Gibbs States, Self-Averaging.

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1. Introduction

The description of infinite-volume equilibrium (Gibbs) states of the models in classical statistical mechanics makes use of the general notion of limiting Gibbs measures (states). These were characterized quite generally for the systems with "bona-fide" interactions as measures by *Dobrushin-Lanford-Ruelle* (DLR) equations, see, e.g. [1]–[5] and Appendix A. For such systems the whole set of limiting Gibbs measures coincides with the closed convex hull of the set of all weak limits of finite-volume Gibbs measures (first introduced in [6, 7]) subjected by local specifications to various boundary conditions. On the other hand, for models of the mean-field type (like the Curie-Weiss ferromagnet), where the interaction depends on the volume and where there is no notion such as interaction in the infinite system, one cannot define limiting Gibbs measures via DLR equations. Therefore, to construct the infinite-volume equilibrium states in this case, one has to exploit weak limits of finite-volume Gibbs measures or Kolmogorov's existence theorem, see [5] and [8]–[10].

For finite systems their corresponding (unique) finite-volume Gibbs measures are prescribed by the *Gibbs ansatz* (see Section 2 and Appendix A). This uniqueness manifests the absence of any phase transitions in a finite volume. The interest in taking the infinite-volume limit has been motivated by understanding of one-to-one correspondence between different limiting states and different phases.

The description of *all* limiting Gibbs states for a nontrivial model and arbitrary temperatures (for high temperatures, as a rule, the uniqueness theorem can be proved) is a rather difficult problem. For example, this has been solved for the 2-D Ising model [11], but not for D=3. A problem arising in the last case is that, for the low temperature region, besides the two well-known translation-invariant ferromagnetic states there are a lot of nontranslation-invariant limiting Gibbs states (Dobrushin phases) [12, 13]. Therefore, for the models exploited in Statistical Physics, a knowledge of the structure of limiting Gibbs states is of great interest. The mean-field models are among the most popular ones which come immediately to mind.

In the case of a homogeneous external field a set of all limiting Gibbs states was constructed in [8] for ferromagnetic Ising- and in [9, 10] for ferromagnetic n-vector Curie-Weiss models by means of a generalized *quasi-average method.*

As indicated in [14] and recently in more general framework, in [15, 16], the Størmer's de Finetti-theorem [17] allows one to characterize the infinite-volume states by the *Gibbs variational principle* for a very general class of *homogeneous* (quantum) mean-field systems.

Recently there has been a considerable interest in the rigorous study of the thermodynamics of lattice spin systems in the presence of *frozen-in* random external fields [18]–[24]. This randomness is referred to as *quenched.* In contrast to a homogeneous field, quenched random-field models can manifest a very nontrivial behavior. "Switching on" a homogeneous external field in the ferromagnetic Ising model ($D \geq 2$) is known, see, e.g., [4, 5, 25], to suppress the symmetry breaking phase transition. On the contrary, for the ferromagnetic *random-field Ising model* (RFIM) in dimensions $D \geq 3$ a first-order phase transition with symmetry breaking persists at a weak enough disorder. This statement has been rigorously established in [21, 22]. Recently, the *rounding effect* of a first-order phase transition in quenched disordered systems has been discovered in [24]. Specific implications are found in RFIM. By refined arguments it is rigorously shown that for $D \leq 2$ an *arbitrary weak* quenched random field suppresses the first-order phase transition. For all temperatures the *infinite-volume Gibbs state* (IVGS), for $D \leq 2$ RFIM, is *unique* for *almost all* (a.a.) field configurations.

The aim of the present paper is to elucidate the problem of construction and description of the set of IVGS for the Curie-Weiss version of the ferromagnetic RFIM. Though Curie-Weiss RFIM has been studied from different points of view (see, e.g., [9, 10] and [26]), including the problems of self-averaging and fluctuations [19, 27, 28], as far as we know, no complete constructive description of all its IVGS exists.

Here we develop our approach to IVGS for the Curie-Weiss RFIM started in [29, 30], stressing that besides the random external field these states depend on an additional random parameter. A particular manifestation of this additional randomness is violation of the *self-averaging property*, e.g., for magnetization.

In [29] we proposed the notion of *conditional self-averaging* to cover this case.

Definition 1.1 *Let $\{\phi_n\}_{n\geq 1}$ be a sequence of random variables defined on a probability space $(\mathcal{R}, \mathcal{B}(\mathcal{R}), \lambda)$. We say that this sequence is* partially (conditionally) self-averaging *if there exists a sequence of finite partitions $\left\{\mathcal{D}_n = \{D_i^{(n)}\}_{i=1}^k\right\}_{n\geq 1}$ of the space $\mathcal{R}$ such that for conditional expectations $\{\mathbf{E}(\phi_n|\mathcal{D}_n)\}_{n\geq 1}$ we have the following convergence in probability λ:*

$$\lim_{n\to\infty} (\phi_n - \mathbf{E}(\phi_n|\mathcal{D}_n)) \stackrel{\lambda}{=} 0$$

and $\lim_{n\to\infty} \lambda\left(D_i^{(n)}\right)$ exists for each $i = 1, 2, \ldots, k$.

Remark 1.1. The above construction is equivalent (see, e.g., [31]) to the following: there is a random variable ϕ defined on a probability space $(\mathcal{R}', \mathcal{B}(\mathcal{R}'), \lambda')$ such that $\phi_n \xrightarrow{d} \phi$ in distribution as $n \to \infty$;

(ii) there exists a finite partition $\mathcal{D}_\phi = \{D_1, D_2, \ldots, D_k\}$ of $\mathcal{R}'$, generated by the random variable ϕ, i.e., $\phi(\cdot) = \sum_{i=1}^k y_i I_{D_i}(\cdot)$, where $I_D(\cdot)$ is indicator of the event D and $\cup_{i=1}^k D_i = \mathcal{R}'$.

If this partition is trivial, $k = 1$ and $\mathcal{R} = \mathcal{R}'$, then one gets the standard self-averaging [19, 20]. In this case d-convergence in *(i)* is equivalent to the convergence in probability λ and one can often prove the convergence with $\Pr = 1$, i.e., λ-almost sure (λ-a.s.), see, e.g. [18]–[20].

The main thesis developed in the present paper is the following: the IVGS for the Curie-Weiss RFIM are *random measures* (on the space of infinite spin configurations with corresponding σ-algebra of measurable subsets) which, as random elements, are defined on an appropriate probability space, see Section 2. There we introduce a general concept of random IVGS and propose two definitions of them (see Definition 2.1 and 2.2) relevant to the problem. Two points are important there. The first is the view on the IVGS as accumulation points of the sequences of finite-volume Gibbs measures. The second is to consider the *random* IVGS *as a limit* of the *random* finite-volume Gibbs measures corresponding to a sequence of the RFIM Hamiltonians. In Section 3 we illustrate the relevance of our Definition 2.2 to the case of random IVGS for the Curie-Weiss RFIM. The *regularity condition* that is the key to the correctness of our construction (see Theorem 3.3) resembles the *consistency condition* by Aizenman and Wehr [24]. The structure of the random IVGS is analyzed in Section 4. Using the explicit representation for the finite-volume Gibbs state and the Laplace method modified to accept a random external field (Appendix B) we show that the random IVGS for the Curie-Weiss RFIM are *random mixtures* of the *pure states* which are infinite product-measures corresponding to the *free Ising model* in homogeneous fields. We show there also that by the (generalized) *quasi-average method* [8, 9] one can change the distribution of the coefficients in these random mixtures. In Section 5 we consider a particular example when the external field is a stationary sequence of dichotomous random variables. The thermodynamics of this model was investigated in [26]. We consider here two points: the evolution of the random IVGS (we compare them also with the set of accumulation points) when the system crosses the critical line on the phase diagram and the *conditional self-averaging* nature of the random magnetization in this case. In Appendix A we collect some basic definitions and properties of the probability measures on the space of the infinite Ising spin configurations (limiting Gibbs measures). The Laplace method for the case of Curie-Weiss RFIM is presented in the expositive Appendix B.

2. Setup and Statement of the Problem

Let $(\Omega, \mathcal{A}, p)$ be a probability space and let $\mathbf{h}(\cdot) \equiv \{h_j(\cdot)\}_{j\in\mathbf{Z}}$ be a sequence of *independent identically distributed* ($\mathbf{R}$-valued) *random variables* (i.i.d.r.v.) defined on this space. Here $\mathbf{Z}$ is an arbitrary integer lattice and the distributions $F_{h_j}(x) \equiv p\{\omega : h_j(\omega) \le x\}$ are identical for all $j \in \mathbf{Z}$. Then the probability space $(\mathbf{R}^{\mathbf{Z}}, \mathcal{F}, \lambda)$ with the Borel σ-algebra $\mathcal{F} = \mathcal{B}(\mathbf{R}^{\mathbf{Z}})$ and the infinite product measure $d\lambda = \prod_{j\in\mathbf{Z}} d\nu_j$ with identical one-dimensional marginals $d\nu_j(x) = F_{h_j}(dx)$ corresponds to the *random field* of configurations $\mathbf{h}(\cdot) : \Omega \to \mathbf{R}^{\mathbf{Z}}$. Below we shall denote by $\mathbf{h}(\omega)$ (or simply by $\mathbf{h}$ for short) a realization of the random field $\mathbf{h}(\cdot)$, which corresponds to $\omega \in \Omega$.

Let $\Lambda \subset \mathbf{Z}$ be a finite subset with cardinality $|\Lambda| = N$. Then the *free* Ising spin system in the external field $\mathbf{h}(\omega)$, $\omega \in \Omega$, is defined by the Hamiltonian

$$H_\Lambda^{(0)}(\mathbf{s}^\Lambda; \mathbf{h}(\omega)) = -\sum_{j\in\Lambda} h_j(\omega) s_j, \quad s_j = \pm 1, \tag{2.1}$$

where $\mathbf{s}^\Lambda = \{s_j\}_{j\in\Lambda} \in \mathcal{S}^\Lambda \equiv \{-1; +1\}^\Lambda$. Hence, by the definition of the Gibbs state (see Appendix A, Eq. (A.1)) the finite-volume free Gibbs measure for a fixed configuration $\mathbf{h}(\omega)$ has the form

$$P_{\Lambda}^{(0)}(\mathbf{s}^{\Lambda};\mathbf{h}(\omega)) = \prod_{j\in\Lambda} \frac{\exp(\beta s_j h_j(\omega))}{2\cosh\beta h_j(\omega)}. \tag{2.2}$$

The Hamiltonian (2.1) and the measure (2.2) for the free RFIM are independent of the external spin configurations $\mathbf{s}^{\overline{\Lambda}}$, cf. (A.1). The Curie-Weiss RFIM corresponds to the perturbation of (2.1) by the *Curie-Weiss* (CW) *ferromagnetic interaction:*

$$H_{\Lambda}(\mathbf{s}^{\Lambda};\mathbf{h}(\omega)) = -\frac{1}{2N}\sum_{i,j\in\Lambda} s_i s_j - \sum_{j\in\Lambda} h_j(\omega) s_j. \tag{2.3}$$

Here we impose the *empty external conditions*, otherwise the Hamiltonian (2.3) is ill-defined because of the infinite-range interaction.

To proceed to the thermodynamics of models (2.1),(2.3) one has to specify the randomness. The equilibrium properties of the system concern the *free-energy density* $f(\beta;\mathbf{h}(\omega))$ in the thermodynamic limit, $t\text{-}\lim(\cdot) \equiv \lim\limits_{\Lambda\uparrow\mathbf{Z}}$:

$$f(\beta;\mathbf{h}(\omega)) = t\text{-}\lim\left[-\frac{1}{\beta N}\ln\sum_{\mathbf{s}^{\Lambda}\in\mathcal{S}^{\Lambda}}\exp\left(-\beta H_{\Lambda}(\mathbf{s}^{\Lambda};\mathbf{h}(\omega))\right)\right], \tag{2.4}$$

and *infinite-volume Gibbs states* corresponding to *typical configurations* of the external field $\mathbf{h}(\cdot)$. Since we consider the free-energy density and the Gibbs states separately for each configuration of the random field $\mathbf{h}(\cdot)$, the free and the Curie-Weiss RFIM are systems with *quenched randomness* [19]–[24].

By the *Strong Law of Large Numbers* (SLLN), see, e.g. [31], the free-energy density (2.4) for the free RFIM (2.1) is λ-almost surely (λ-a.s. or with Pr $= 1$) independent of the fixed configuration $\mathbf{h}(\omega)$ (*self-averaging*)

$$f^{(0)}(\beta;\mathbf{h}(\cdot)) \overset{\lambda\text{-a.s.}}{=} -\beta^{-1}\int_{\mathbf{R}^1} F_h(dx)\ln(2\cosh\beta x). \tag{2.5}$$

To construct for (2.1) the infinite-volume Gibbs states we can follow the standard scheme outlined in Appendix A.

Let $\bar{s}^{\overline{\Lambda}} \equiv \bar{s}|\overline{\Lambda} \equiv \{\bar{s}_i\}_{i\in\overline{\Lambda}}$; $\overline{\Lambda} = \mathbf{Z}\setminus\Lambda$, be a spin configuration outside Λ and $\mathcal{S}_{\overline{\Lambda}}(\bar{\mathbf{s}})$ be the set of infinite spin configurations coinciding with a fixed $\bar{s}^{\overline{\Lambda}}$ for $i\in\overline{\Lambda}$. Then the extension of the free measure (2.2) to the Borel σ-algebra $\mathcal{B}(\mathcal{S})$ is (see Appendix A)

$$\overline{P}_{\Lambda,\bar{s}}^{(0)}(A;\mathbf{h}(\omega)) = \sum_{\mathbf{s}^{\Lambda}\in\pi_{\Lambda}(A\cap\mathcal{S}_{\overline{\Lambda}}(\bar{\mathbf{s}}))} P_{\Lambda}^{(0)}(\mathbf{s}^{\Lambda};\mathbf{h}(\omega)),\quad A\in\mathcal{B}(\mathcal{S}). \tag{2.6}$$

Here π_{Λ}: $\mathbf{s}\to\mathbf{s}^{\Lambda}$. Let $C_I(B)$ be a cylinder set with support I and base $B\in\mathcal{S}^I$. Then by (2.2) and (2.6) one finds that for $C_I(B)\in\mathbf{C}(\mathcal{S})$, $\mathcal{S}\equiv\mathcal{S}^{\mathbf{Z}}$,

$$\overline{P}_{\Lambda,\bar{s}}^{(0)}(C_I(B);\mathbf{h}(\omega)) = \sum_{\mathbf{s}^I\in B}\prod_{j\in I}\frac{\exp(\beta s_j h_j(\omega))}{2\cosh\beta h_j(\omega)} \equiv P^{(0)}(C_I(B);\mathbf{h}(\omega)) \tag{2.7}$$

is independent of $\overline{\mathbf{s}}^{\overline{\Lambda}}$ for all $\Lambda \supseteq I$. Therefore, by Proposition A.1 the probability measure $P^{(0)}(\cdot;\mathbf{h}(\omega))$ is a *unique weak limit* of the sequence $\left\{\overline{P}^{(0)}_{\Lambda,\overline{\mathbf{s}}}(\cdot;\mathbf{h}(\omega))\right\}_{\Lambda\subset\mathbf{Z}}$ when $\Lambda \uparrow \mathbf{Z}$, for any fixed configuration $\mathbf{h}(\omega)$. Hence, by Definition A.1 for each fixed configuration $\mathbf{h}(\omega)$ the probability measure $P^{(0)}(\cdot;\mathbf{h}(\omega))$ on $\mathcal{B}(\mathcal{S})$ is *unique infinite-volume Gibbs state* corresponding to the free Ising spin system (2.1).

Remark 2.1. Let the Hamiltonian $H_\Lambda(\mathbf{s}^\Lambda;\mathbf{h}(\omega))$ be independent of the configuration outside Λ. Then, in spite of explicit dependence of the extension $\overline{P}_{\Lambda,\overline{\mathbf{s}}}(\cdot;\mathbf{h}(\omega))$ on $\overline{\mathbf{s}}^{\overline{\Lambda}}$, the limiting measure $P(\cdot;\mathbf{h}(\omega)) = t\text{-}\lim \overline{P}_{\Lambda,\overline{\mathbf{s}}}(\cdot;\mathbf{h}(\omega))$ is independent of configuration $\overline{\mathbf{s}}$. To verify this, notice that for any cylinder set $C_I(B)$ with support $I \subset \Lambda$ one gets $\pi_\Lambda\left\{C_I(B)\cap \mathcal{S}_{\overline{\Lambda}}(\overline{\mathbf{s}})\right\} = \pi_\Lambda(C_I(B))$. Therefore, $\overline{P}_{\Lambda,\overline{\mathbf{s}}}\left(C_I(B);\mathbf{h}(\omega)\right) = \sum_{\mathbf{s}^\Lambda\in\pi_\Lambda(C_I(B))} P_\Lambda(\mathbf{s}^\Lambda;\mathbf{h}(\omega))$, cf. (2.6), is independent of $\overline{\mathbf{s}}$. By Proposition A.1 measure $P(\cdot;\mathbf{h}(\omega))$ on the measurable space $(\mathcal{S},\mathcal{B}(\mathcal{S}))$ is *uniquely* defined by its values on the cylinder sets: $\left\{\text{t-}\lim \overline{P}_{\Lambda,\overline{\mathbf{s}}}\left(C_I(B);\mathbf{h}(\omega)\right)\right\}_{I\subset\mathbf{Z},B\in\mathcal{S}^I}$.

It is clear that we obtain the same result if we follow the line of reasoning of Proposition A.2. The family of marginals

$$\rho^{(0)}_\Gamma(B;\mathbf{h}(\omega)) = \lim_{\Lambda\uparrow\mathbf{Z}} \overline{P}^{(0)}_{\Lambda,\overline{\mathbf{s}}}(B\times\mathcal{S}^{\Lambda\setminus\Gamma};\mathbf{h}(\omega)) = P^{(0)}(\pi_\Gamma^{-1}(B);\mathbf{h}(\omega)), \tag{2.8}$$

$B \in \mathcal{B}(\mathcal{S}^\Gamma)$, $|\Gamma| < \infty$, satisfies the conditions of Proposition A.2 and the limit (2.8) is unique for a fixed configuration $\mathbf{h}(\omega)$. The corresponding IVGS is again a unique infinite product measure $P^{(0)}(\cdot;\mathbf{h}(\omega))$, see (2.7), indexed by configurations of the random field $\mathbf{h}(\cdot)$.

The above observation could motivate the following generalization of Definition A.1 for system with Hamiltonians depending on random parameters, e.g., on the random external field $\mathbf{h}(\cdot)$ defined on probability space $(\Omega,\mathcal{A},p)$.

Definition 2.1 *If for a fixed boundary condition* $\overline{\mathbf{s}}$ *and for p-almost all* ω *the sequence of probability measures* $\left\{\overline{P}_{\Lambda_n,\overline{\mathbf{s}}}(\cdot;\mathbf{h}(\omega))\right\}_{n\geq 1}$ *on* $\mathcal{B}(\mathcal{S})$ *has an unique weak accumulation point* $P_{\overline{\mathbf{s}}}(\cdot;\mathbf{h}(\omega))$, *then we call the random measure* $P_{\overline{\mathbf{s}}}(\cdot;\mathbf{h}(\cdot))$ *a random infinite-volume* Gibbs state *for the random Hamiltonians* $H_{\Lambda,\overline{\mathbf{s}}}(\mathbf{s}^\Lambda;\mathbf{h}(\cdot))$.

If there is a set $A \subset \Omega$, $p(A) > 0$, such that the sequence $\left\{\overline{P}_{\Lambda,\overline{\mathbf{s}}}(\cdot;\mathbf{h}(\omega))\right\}_\Lambda$, for $\omega \in A$, has *more than one* (weak) accumulation point, then we encounter difficulties in interpreting them as realizations of some *random* IVGS, i.e., a measure-valued random element defined on the probability space $(\Omega,\mathcal{A},p)$.

Remark 2.2. Let the DLR equation for the random Hamiltonians $\left\{H_{\Lambda,\overline{\mathbf{s}}}(\mathbf{s}^\Lambda;\mathbf{h}(\cdot))\right\}_\Lambda$ (see (A.2)) have sense for p-almost all (p-a.a.) $\omega \in \Omega$. Then we could resolve the above mentioned difficulties by restricting consideration to some *sufficient subset* of boundary conditions $\overline{\mathbf{s}}$. Let the DLR equation have solutions $\{P_\alpha(\cdot;\mathbf{h}(\omega))\}_{\alpha\in A(\omega)}$ for p-a.a. $\omega \in \Omega$ and there exists the set $\{\overline{\mathbf{s}}_\mu\}_{\mu\in M}$ of boundary conditions (*sufficient subset*) that the set of the weak accumulation points of the sequence $\left\{\overline{P}_{\Lambda,\overline{\mathbf{s}}}(\cdot;\mathbf{h}(\omega))\right\}_{\Lambda\subset\mathbf{Z}}$ reduces to the *unique* measure $P_{\overline{\mathbf{s}}}(\cdot;\mathbf{h}(\omega))$ for any

$\overline{s} \in \{\overline{s}_\mu\}_{\mu \in M}$ and

$$\left\{P_{\overline{s}_\mu}(\cdot;\mathbf{h}(\omega))\right\}_{\mu\in M} = \{P_\alpha(\cdot;\mathbf{h}(\omega))\}_{\alpha\in A(\omega)},$$

for p-a.a. $\omega \in \Omega$. Then for any boundary condition $\overline{s}_\mu$, $\mu \in M$, we can define a random IVGS as measure-valued random element $P_{\overline{s}_\mu}(\cdot;\mathbf{h}(\omega))$, see Definition 2.1. If card $M > 1$, then the random infinite-volume Gibbs state is *non-unique*.

But for the CW RFIM (2.3) the situation is even worse: we cannot use the DLR equation in this case (see Remark A.1) and the only reasonable boundary condition for this model has to be *empty*, i.e., *no spins outside* the set Λ. Formally this corresponds to $\{s_i = 0\}_{i\in\overline{\Lambda}}$, see (2.3). The above observations motivate the proposal of another construction defining IVGS for random spin systems, in particular, for RFIM.

Definition 2.2 *Suppose that for any cylinder set $C \in \mathbf{C}(\mathcal{S})$ random variables $\left\{\overline{P}_{\Lambda,\overline{s}}(C;\mathbf{h}(\cdot))\right\}_\Lambda$ on $(\Omega,\mathcal{A},p)$ converge (for $\Lambda \uparrow \mathbf{Z}$) in some* o*-sense to the random variable $P_{\overline{s}}(C;\cdot)$ on $(X,\mathcal{B}(X),r)$. If for r-a.a. $\chi \in X$ there exists a probability measure $P(\cdot;\chi)$ on $\mathcal{B}(\mathcal{S})$ such that $P(C;\chi) = P_{\overline{s}}(C;\chi)$ for any $C \in \mathbf{C}(\mathcal{S})$, then the random measure $P(\cdot;\cdot)$ we call a* random infinite-volume Gibbs state, *corresponding to the family of the finite-volume Gibbs states $\left\{\overline{P}_{\Lambda,\overline{s}}(\cdot;\mathbf{h}(\cdot))\right\}_\Lambda$.*

Here we have to detail the boundary conditions denoted by $\overline{s}$. For CW ferromagnet they are "empty"and we shall drop $\overline{s}$. The second point is to specify the o-sense. E.g. for the free RFIM o-sense means p-a.s. (even for all $\omega \in \Omega$). Then $\Omega = X$, a (unique) probability measure $P(\cdot;\omega)$ exists and Definition 2.2 gives the same as Definition 2.1.

Below we demonstrate the use of Definition 2.2 for CW RFIM when the existence of the measure $P(\cdot;\chi)$ can be verified by the Kolmogorov theorem.

3. Curie-Weiss Random Field Ising Model

For any fixed $\omega \in \Omega$ by the standard linearization trick [32] the *finite-volume Gibbs state* for the Hamiltonian (2.3) and empty boundary condition can be expressed as (cf. (2.2))

$$P_\Lambda(\mathbf{s}^\Lambda;\mathbf{h}(\omega)) = \int_{\mathbf{R}^1} \mu_\Lambda(dy;\mathbf{h}(\omega))P_\Lambda^{(0)}(\mathbf{s}^\Lambda;\mathbf{h}(\omega)+y). \tag{3.1}$$

Here $(\mathbf{h}(\omega)+y)_{j\in\mathbf{Z}} = h_j(\omega)+y$ and $\mu_\Lambda(dy;\mathbf{h}(\omega))$ is a probability measure with density

$$\frac{\mu_\Lambda(dy;\mathbf{h}(\omega))}{dy} = \frac{\exp\left[-\beta N G_\Lambda(y;\mathbf{h}(\omega))\right]}{\int_{\mathbf{R}^1} dy \exp\left[-\beta N G_\Lambda(y;\mathbf{h}(\omega))\right]}, \tag{3.2}$$

where

$$G_\Lambda(y;\mathbf{h}(\omega)) = \frac{1}{2}y^2 - \frac{1}{\beta N}\sum_{j\in\Lambda} \ln\cosh\left[\beta(h_j(\omega)+y)\right]. \tag{3.3}$$

Now, using (2.6),(2.7) and the explicit expression (3.1) we can extend the finite-volume Gibbs measure (3.1) to the Borel σ-algebra $\mathcal{B}(\mathcal{S})$:

$$\overline{P}_\Lambda(A;\mathbf{h}(\omega)) = \int_{\mathbf{R}^1} \mu_\Lambda(dy;\mathbf{h}(\omega))\overline{P}^{(0)}_{\Lambda,\bar{\mathbf{s}}}(A;\mathbf{h}(\omega)+y), \quad A \in \mathcal{B}(\mathcal{S}) \tag{3.4}$$

Here $\bar{\mathbf{s}}$ is any infinite configuration. (As it has been already mentioned in Remark 2.1, in our case limiting measure $P(\cdot;\mathbf{h}(\omega))$ is independent of extension.) For the measure of any cylinder set $C_I(B)$, $I \subseteq \Lambda, B \subseteq S^I$ one gets (cf. (2.7))

$$\overline{P}_\Lambda(C_I(B);\mathbf{h}) = \int_{\mathbf{R}^1} \mu_\Lambda(dy;\mathbf{h})P^{(0)}(C_I(B);\mathbf{h}+y). \tag{3.5}$$

By compactness arguments (see Proposition A.1) for any fixed configuration $\mathbf{h}$ and $\Lambda \uparrow \mathbf{Z}$, there is at least one subsequence $\left\{\overline{P}_{\Lambda_\alpha}\right\}_{\Lambda_\alpha \uparrow \mathbf{Z}}$ such that $\overline{P}_{\Lambda_\alpha} \Rightarrow P_\alpha$.

To describe the limiting measures (infinite-volume Gibbs states) $\{P_\alpha\}_\alpha$ in an explicit way we need the following statement.

Theorem 3.1 *Let the probability measure $d\nu$, for the quenched random external field $\mathbf{h}$ (see Section 2), be such that $\int_{\mathbf{R}^1} d\nu(x)\,|x| < \infty$. Then for λ-a.a. configurations $\mathbf{h}$ there are subsequences $\{\Lambda_\alpha(\mathbf{h}) \uparrow \mathbf{Z}\}_\alpha$, such that $\mu_{\Lambda_\alpha(h)}(dy;\mathbf{h}) \Rightarrow \mu_\alpha(dy;\mathbf{h})$ and, λ-a.s.,* $\mathrm{supp}\mu_\alpha \subseteq \mathcal{M}(\nu,\beta) = \left\{y_0 \in \mathbf{R} : \min_{y\in\mathbf{R}} G(y) = G(y_0)\right\}$, *where (cf.(3.3))*

$$G(y) = \frac{1}{2}y^2 - \frac{1}{\beta}\int_{\mathbf{R}^1} d\nu(x)\,\ln\cosh[\beta(x+y)]. \tag{3.6}$$

Proof. By Lemma B.1, for $\Lambda \uparrow \mathbf{Z}$ one gets $G_\Lambda(y;\mathbf{h}) \xrightarrow{\lambda\text{-a.s.}} G(y)$ and this convergence is locally uniform in $y \in \mathbf{R}$. Hence, by definition (3.2) and Eq. (3.6), for any $\varepsilon > 0$ there exists compact $K \subset \mathbf{R}$ such that for λ-a.a. configurations $\mathbf{h}$ and all large enough $\Lambda \subset \mathbf{Z}$ we have $\mu_\Lambda(\mathbf{R}\setminus K;\mathbf{h}) < \varepsilon$. Then the first assertion of the theorem is a consequence of Prohorov's compactness theorem [31] for each $\mathbf{h}$ from the above mentioned set of the λ-a.a. configurations. According to Corollary B.2 we have $\int_{\mathbf{R}^1} \mu_{\Lambda_\alpha(h)}(dy;\mathbf{h})g(y) \xrightarrow{\lambda\text{-a.s.}} 0$ for any continuous function g with a compact support such that $\mathrm{supp}\, g \cap \mathcal{M}(\nu,\beta) = \{\emptyset\}$. Thus, any accumulation point $\mu_\alpha(dy;\mathbf{h})$ of the sequence $\{\mu_\Lambda(dy;\mathbf{h})\}_\Lambda$ has support in $\mathcal{M}(\nu,\beta)$. ■

Corollary 3.1 *For λ-a.a. configurations of the quenched field $\mathbf{h}$ the set of infinite-volume Gibbs states $\{P_\alpha\}_\alpha$ for the Curie-Weiss RFIM is non-empty and they are* quasi-free, *i.e., (linear convex) superpositions of the* shifted free states, *cf. (3.5),*

$$P_\alpha(A;\mathbf{h}) = \int_{\mathbf{R}^1} \mu_\alpha(dy;\mathbf{h})P^{(0)}(A;\mathbf{h}+y), \quad A \in \mathcal{B}(\mathcal{S}), \tag{3.7}$$

where weak limits $\{\mu_\alpha(dy;\mathbf{h})\}_\alpha$ are defined by Theorem 3.1.

Proof. Let for a fixed $\mathbf{h}$ from the λ-a.a. configurations of Theorem 3.1 $\overline{P}_{\Lambda_\alpha(h)} \Rightarrow P'_\alpha$. Then by this theorem there is a subsequence $\left\{\mu_{\Lambda_\gamma}\right\}_{\Lambda_\gamma}$ (where $\{\Lambda_\gamma\}$ is a subsequence of $\{\Lambda_\alpha(\mathbf{h})\}$) such that $\mu_{\Lambda_\gamma} \Rightarrow \mu_\alpha$. For any cylinder set $C \in \mathbf{C}(\mathcal{S})$ there is a large enough $\tilde{\Lambda}_\gamma \supset \operatorname{supp} C$ that for all $\Lambda_\gamma(\supset \tilde{\Lambda}_\gamma) \uparrow \mathbf{Z}$ we can use (3.5). Hence, by the weak convergence of the sequence $\left\{\mu_{\Lambda_\gamma}\right\}_{\Lambda_\gamma \supset \tilde{\Lambda}_\gamma}$ one gets $\overline{P}_{\Lambda_\gamma}(C;\mathbf{h}) \to P_\alpha(C;\mathbf{h})$, where P_α is the quasi-free state defined by (3.7) and consequently $\overline{P}_{\Lambda_\alpha(\mathbf{h})}(C;\mathbf{h}) \to P_\alpha(C;\mathbf{h})$. The last observation, together with Proposition A.1, completes the proof for any Borel set $A \in \mathcal{B}(\mathcal{S})$. ■

Remark 3.1. Let $\mathcal{M}(\nu,\beta) = \{y_0(\beta)\}$, i.e., the function (3.6) has a unique minimum. Then for λ-a.a. configurations $\mathbf{h}$ the sequences $\{\mu_\Lambda(dy;\mathbf{h})\}_\Lambda$ have unique (non-random) accumulation point $\mu_\alpha(dy;\mathbf{h}) = \delta(y - y_0(\beta))dy$. Therefore, in this case we get for the Curie-Weiss RFIM the (unique) random IVGS (see Definitions 2.1 and 2.2) which, according to (3.7), is *"shifted" free state*:

$$P_\alpha(A;\mathbf{h}) = P^{(0)}(A;\mathbf{h}+y_0), \quad A \in \mathcal{B}(\mathcal{S}). \tag{3.8}$$

Remark 3.2. Let, for instance, $\mathcal{M}(\nu,\beta) = \{y_{01}(\beta), y_{02}(\beta)\}$, see Section 4 and example in Section 5. Then Theorem 3.1 describes a general structure of the weak accumulation points $\{\mu_\alpha(dy;\mathbf{h})\}_\alpha$ of the sequence $\{\mu_\Lambda(dy;\mathbf{h})\}_\Lambda$ for λ-a.a. $\mathbf{h}$:

$$\mu_\alpha(dy;\mathbf{h}) = w\text{-}\lim_{\Lambda_\alpha(h)\uparrow\mathbf{Z}} \mu_{\Lambda_\alpha(h)}(dy;\mathbf{h}) = [t_\alpha\delta(y-y_{01}) + (1-t_\alpha)\delta(y-y_{02})]\, dy. \tag{3.9}$$

Here the coefficients $t_\alpha \in [0,1]$ depend, in general, on the configuration $\mathbf{h}(\omega)$ and the particular choice of subsequence $\{\Lambda_\alpha(h)\}$ for a fixed $\mathbf{h}(\omega)$.

Therefore, it is important to investigate the relevance of Definition 2.2 for the case when the set $\mathcal{M}(\nu,\beta)$ contains more then one point. By Remark 3.2 it could be anticipated that IVGS for the Curie-Weiss RFIM are a "random mixtures" of the shifted free states (3.8). Below we elucidate this point in the frame of Definition 2.2.

First, suppose that the o-sense, for convergence of random variables in Definition 2.2, coincides with $\overline{P}_\Lambda(C;\mathbf{h}(\cdot)) \xrightarrow{p} \tilde{P}(C;\cdot)$, $C \in \mathbf{C}(\mathcal{S})$, as $\Lambda \uparrow \mathbf{Z}$, in *probability* on the common probability space $(\Omega, \mathcal{A}, p) = (X, \mathcal{B}(X), r)$. Here $\xi_n \xrightarrow{p} \xi$ means that $\lim_{n\to\infty} p\{\omega \in \Omega : |\xi_n(\omega) - \xi(\omega)| > \varepsilon\} = 0$ for any $\varepsilon > 0$, see, e.g. [31]. Then it is easy to verify that the random variables $\rho_\Gamma(B;\mathbf{h}(\cdot)) \equiv \lim_{\Lambda\uparrow\mathbf{Z}} P_\Lambda(\pi_\Lambda \circ \pi_\Gamma^{-1}(B);\mathbf{h}(\cdot))$, $\Gamma \subset \Lambda$, $B \in \mathcal{B}(\mathcal{S}^\Gamma)$, make up a family of consistent marginals for p-a.a. $\omega \in \Omega$. Hence, by Proposition A.2 (Kolmogorov's theorem) for p-a.a. $\omega \in \Omega$ they generate a unique probability measure $P(\cdot;\mathbf{h}(\omega))$ on $(\mathcal{S};\mathcal{B}(\mathcal{S}))$. So, by Definition 2.2 the random measure $P(\cdot;\mathbf{h}(\cdot))$ is a random IVGS. The next statement gives sufficient conditions for convergence of the random variables $\left\{\overline{P}_\Lambda(C;\mathbf{h}(\cdot))\right\}_\Lambda, C \in \mathbf{C}(\mathcal{S})$, in probability.

Theorem 3.2 *If for the sequence of random measures $\{\mu_\Lambda(dy;\mathbf{h}(\cdot))\}_\Lambda$ there exists measure $\mu(dy;\mathbf{h}(\cdot))$ such that the random variables*

$$\int_{\mathbf{R}^1} \mu_\Lambda(dy;\mathbf{h}(\cdot))\varphi(y) \xrightarrow{p} \int_{\mathbf{R}^1} \mu(dy;\mathbf{h}(\cdot))\varphi(y), \tag{3.10}$$

as $\Lambda \uparrow \mathbf{Z}$, *for any continuous function* $\varphi \in C(\mathbf{R})$, *then*

$$\overline{P}_\Lambda(C;\mathbf{h}(\cdot)) \xrightarrow{p} P(C;\mathbf{h}(\cdot)) \equiv \tilde{P}(C;\cdot),\ C \in \mathbf{C}(\mathcal{S}), \tag{3.11}$$

where random probability measure

$$P(\cdot;\mathbf{h}(\cdot)) = \int_{\mathbf{R}^1} \mu(dy;\mathbf{h}(\cdot))P^{(0)}(\cdot;\mathbf{h}(\cdot)+y).$$

Proof. According to (2.7) $\varphi(y) \equiv P^{(0)}(C;\mathbf{h}(\omega)+y) \in C(\mathbf{R})$ for any $C \in \mathbf{C}(\mathcal{S})$ and $\omega \in \Omega$. Then by condition (3.10) and the explicit formula (3.5) one gets (3.11) for $C \in \mathbf{C}(\mathcal{S})$. The last step is extension of the measure (3.11) from the cylinder sets $\mathbf{C}(\mathcal{S})$ to the probability measure on $\mathcal{B}(\mathcal{S})$. ■

Remark 3.3. Now, let the o-sense in Definition 2.2 coincide with

$$\overline{P}_\Lambda(C;\mathbf{h}(\cdot)) \xrightarrow{d} \tilde{P}(C;\cdot),\ C \in \mathbf{C}(\mathcal{S}),$$

as $\Lambda \uparrow \mathbf{Z}$, in *distribution*. Here $\xi_n \xrightarrow{d} \xi$ means that $\lim_{n\to\infty} \mathbf{E}f(\xi_n) = \mathbf{E}f(\xi)$ for any bounded continuous function $f(\cdot)$, see, e.g. [31]. Then, in general, $(\Omega, \mathcal{A}, p) \neq (X, \mathcal{B}(X), r)$ and $\overline{P}_\Lambda(C;\mathbf{h}(\cdot)) \xrightarrow{d} \tilde{P}(C;\cdot)$ means that $\tilde{P}(C;\cdot)$ is a *symbol (representative)* of the *family* $\left\{\tilde{P}_\zeta(C;\cdot)\right\}_\zeta$ of all random variables with the *same* distribution function $r\left\{\tilde{P}_\zeta(C;\cdot) \leq x\right\} = r\left\{\tilde{P}(C;\cdot) \leq x\right\}$, $\forall x \in \mathbf{R}$. Therefore, in this case in general, the random variables $\rho_\Gamma(M;\cdot) = d\text{-}\lim_{\Lambda\uparrow\mathbf{Z}} P_\Lambda(\pi_\Lambda \circ \pi_\Gamma^{-1}(M);\mathbf{h}(\cdot))$ do not inherit *additivity* or *consistency* of the finite-volume marginals

$$\left\{P_\Lambda(\pi_\Lambda \circ \pi_\Gamma^{-1}(M);\mathbf{h}(\cdot))\right\}_{\Lambda\supset\Gamma},\ M \in \mathcal{B}(\mathcal{S}^\Gamma).$$

For example, the marginals $\rho_\Gamma(M;\cdot)$ and $\rho_\Delta(M \times \mathcal{S}^{\Delta\setminus\Gamma};\cdot)$, $\Delta \supset \Gamma$, $M \in \mathcal{B}(\mathcal{S}^\Gamma)$, can be realized on $(X, \mathcal{B}(X), r)$ as *independent* random variables.

Suppose there exists measure $\mu(dy;\cdot)$ on $(X, \mathcal{B}(X), r)$ such that for the random measures $\mu_\Lambda(dy;\mathbf{h}(\cdot))$ we have (cf. (3.10))

$$\int_{\mathbf{R}^1} \mu_\Lambda(dy;\mathbf{h}(\cdot))\varphi(y) \xrightarrow{d} \int_{\mathbf{R}^1} \mu(dy;\cdot)\varphi(y), \tag{3.12}$$

as $\Lambda \uparrow \mathbf{Z}$, and $\varphi \in C(\mathbf{R})$. It is clear that if in the family of all possible (weak) d-limits of the sequence $\{\mu_\Lambda\}_\Lambda$ we *fix a representative* $\mu^*(dy;\chi)$, $\chi \in X$, then for this unique and independent of Γ representative the marginals

$$\rho_\Gamma^*(M;\hat{\chi}) = \int_{\mathbf{R}^1} \mu^*(dy;\chi)P^{(0)}(\pi_\Gamma^{-1}(M);\mathbf{h}(\omega)),\quad M \in \mathcal{B}(\mathcal{S}^\Gamma) \tag{3.13}$$

for each fixed $\hat{\chi} \equiv (\chi,\omega) \in X \times \Omega$, are a *consistent family* of the probability measures on $(\mathcal{S}^\Gamma, \mathcal{B}(\mathcal{S}^\Gamma))$, $\Gamma \subset \mathbf{Z}$. Hence, by the Proposition A.2 (Kolmogorov's theorem) there exists a (random) probability measure on $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$ that has the form (cf. (3.11))

$$P_\chi^*(\cdot;\mathbf{h}(\omega)) = \int_{\mathbf{R}^1} \mu^*(dy;\chi)P^{(0)}(\cdot;\mathbf{h}(\omega)),\quad (\chi,\omega) \in X \times \Omega. \tag{3.14}$$

Remark 3.4. If $\int_{\mathbf{R}^1}\mu_\Lambda(dy;\mathbf{h}(\cdot))\varphi(y) - \int_{\mathbf{R}^1}\mu(dy;\cdot)\varphi(y) \xrightarrow{d} 0$, as $\Lambda\uparrow\mathbf{Z}$, for any $\varphi\in C(\mathbf{R})$, then this implies $\int_{\mathbf{R}^1}\mu_\Lambda(dy;\mathbf{h}(\cdot))\varphi(y) \xrightarrow{p} \int_{\mathbf{R}^1}\mu(dy;\cdot)\varphi(y)$. Therefore, to construct the limiting Gibbs states we have to return to Theorem 3.2.

Let $I=\{i_1,i_2,\ldots,i_n\}$ be a finite subset of $\mathbf{Z}$ and $D\left(\overline{\mathbf{h}}_I\right)=\left(\otimes_{k=1}^n\overline{h}_{i_k}\right)\otimes\mathbf{R}^{\mathbf{Z}\setminus I}$ be the subset of configurations $\{h_i\}_{i\in\mathbf{Z}}=\mathbf{h}$ with the fixed realization $\otimes_{k=1}^n\overline{h}_{i_k}$ on I. Then $\mu_\Lambda(dy;\mathbf{h}|\overline{\mathbf{h}}_I)$ is a random measure-valued function on the probability space $(D(\overline{\mathbf{h}}_I),\mathcal{B}(D(\overline{\mathbf{h}}_I)),\prod_{j\in\mathbf{Z}\setminus I}\nu_j)$ defined by restriction to $D(\overline{\mathbf{h}}_I)$: $\mu_\Lambda(\cdot;\mathbf{h}|\overline{\mathbf{h}}_I)=\mu_\Lambda(\cdot;\mathbf{h})|D(\overline{\mathbf{h}}_I)$.

The following statement establishes sufficient conditions for convergence of the random variables $\left\{\overline{P}_\Lambda(C;\mathbf{h}(\cdot))\right\}_\Lambda$, $C\in\mathbf{C}(\mathcal{S})$, in distribution.

Theorem 3.3 *Suppose that for any finite subset $I\subset\mathbf{Z}$ the sequence $\{\mu_\Lambda(dy;\mathbf{h}(\cdot))\}_\Lambda$ satisfies the* regularity condition *for some measure $\mu(dy;\cdot)$ (cf. (3.12)):*

$$\int_{\mathbf{R}^1}\mu_\Lambda(dy;\mathbf{h}|\overline{\mathbf{h}}_I)\varphi(y) \xrightarrow{d} \int_{\mathbf{R}^1}\mu(dy;\chi)\varphi(y),\quad \varphi\in C(\mathbf{R}), \tag{3.15}$$

as $\Lambda\uparrow\mathbf{Z}$. Then for any cylinder set $C\in\mathbf{C}(\mathcal{S})$ and $\Lambda\uparrow\mathbf{Z}$ one gets

$$\overline{P}_\Lambda(C;\mathbf{h}(\cdot)) \xrightarrow{d} P_\chi(C;\mathbf{h}(\cdot)) \equiv \int_{\mathbf{R}^1}\mu(dy;\chi)P^{(0)}(C;\mathbf{h}(\cdot)+y), \tag{3.16}$$

see also (3.14).

Proof. By Eq. (2.7) for any $C\in\mathbf{C}(\mathcal{S})$ we get: $\overline{P}_\Lambda^{(0)}(C;\mathbf{h}(\cdot)+y)=P^{(0)}(C;\mathbf{h}_{I_C}(\cdot)+y)\in C(\mathbf{R})$ if supp$C=I_C\subset\Lambda$. Here $\mathbf{h}_{I_C}\equiv\mathbf{h}|I_C$. Then by Eq. (3.5) $\overline{P}_\Lambda(C;\mathbf{h})=\langle P^{(0)}(C;\mathbf{h}+y)\rangle_{\mu_\Lambda}$, where $<->_{\mu_\Lambda}=\int_{\mathbf{R}^1}\mu_\Lambda(dy;\mathbf{h})(-)$. Hence, by (2.7) $\overline{P}_\Lambda(C;\mathbf{h})=\langle P^{(0)}(C;\mathbf{h}_{I_C}+y)\rangle_{\mu_\Lambda}$ and by the definition of restriction to $D(\mathbf{h}_{I_C})$ one has

$$\langle P^{(0)}(C;\mathbf{h}_{I_C}+y)\rangle_{\mu_\Lambda}=\int_{\mathbf{R}^1}\mu_\Lambda(dy;\mathbf{h}|\mathbf{h}_{I_C})P^{(0)}(C;\mathbf{h}_{I_C}+y). \tag{3.17}$$

Therefore, using the regularity condition (3.15) we get for (3.17) that

$$\langle P^{(0)}(C;\mathbf{h}_{I_C}+y)\rangle_{\mu_\Lambda} \xrightarrow{d} \langle P^{(0)}(C;\mathbf{h}_{I_C}+y)\rangle_{\mu}, \tag{3.18}$$

when $\Lambda\uparrow\mathbf{Z}$. This proves the assertion (3.16) because $P^{(0)}(C;\mathbf{h}_{I_C}+y)=P^{(0)}(C;\mathbf{h}+y)$, see (2.7). ■

In the next section we use the results of Theorems 3.1–3.3 and Remark 3.1 to elucidate the structure of the *random* IVGS for the Curie-Weiss RFIM in the frame of Definition 2.2.

4. Random Infinite-Volume Gibbs States

To get explicit formulas we consider a particular case when card$\mathcal{M}(\nu,\beta)\le 2$, see Remarks 3.1 and 3.2. For card$\mathcal{M}(\nu,\beta)>2$ we encounter difficulties in an explicit description of the structure of the limiting measures o-$\lim_{\Lambda\uparrow\mathbf{Z}}\mu_\Lambda(dy;\mathbf{h})$.

Theorem 4.1 *Let* $\mathbf{h}(\cdot) = \{h_j(\cdot) \in \mathbf{R}^1\}_{j\in\mathbf{Z}}$ *be a sequence of i.i.d.r.v. (corresponding to external (quenched) random field for the Curie-Weiss model (2.3)) defined on the probability space* $(\Omega, \mathcal{A}, p)$ *with* $\nu(x) = p\{\omega \in \Omega : h_j(\omega) \le x\}$ *satisfying* $\int_{\mathbf{R}^1} d\nu(x)\,|x| < \infty$. *Let the function* $G(y)$ *(2.6) be such that* $\mathcal{M}(\nu,\beta) = \{y_{01}, y_{02}\}$ *and* $\partial_y^2 G(y_{0i}) > 0$, $i = 1,2$. *Then the sequence* $\{\mu_\Lambda(dy; \mathbf{h}(\cdot))\}_\Lambda$ *satisfies the regularity condition (3.15), where*

$$\mu(dy;\chi) = \{t(\chi)\delta(y - y_{01}) + (1 - t(\chi))\delta(y - y_{02})\}\, dy. \tag{4.1}$$

Here $t(\cdot) \in \{0,1\}$ *is random variable* $t(\cdot) = \{0,1\}$ *with* $\Pr(t(\cdot) = 0) = \Pr(t(\cdot) = 1) = \frac{1}{2}$, *cf. (3.9).*

We start the proof with the following

Lemma 4.1 *Let* $\mathcal{M}(\nu,\beta) = \{y_{0i}\}_{i=1}^m$ *and* $\partial_y^2 G(y_{0i}) = k_i > 0$ $(i = 1,2,\ldots,m)$. *If for each* $y_{0i} \in \mathcal{M}(\nu,\beta)$ *and for p-a.a.* $\omega \in \Omega$ *there exists the sequence* $\left\{y_{0i}^{(\Lambda)}\right\}_\Lambda$ *such that* $\partial_y G_\Lambda\left(y_{0i}^{\Lambda}; \mathbf{h}(\omega)\right) = 0$ *and* $y_{0i}^{(\Lambda)} \to y_{0i}$, *as* $\Lambda \uparrow \mathbf{Z}$. *Then the random variables*

$$\Delta_\Lambda(i,j;\mathbf{h}(\cdot)) = \beta\sqrt{N}\left[G_\Lambda\left(y_{0i}^{(\Lambda)}; \mathbf{h}(\cdot)\right) - G_\Lambda\left(y_{0j}^{(\Lambda)}; \mathbf{h}(\cdot)\right)\right],\ i,j = 1,2,\ldots,m. \tag{4.2}$$

converge in distribution,

$$\lim_{\Lambda\uparrow\mathbf{Z}} \Delta_\Lambda(i,j;\mathbf{h}(\cdot)) \stackrel{d}{=} \mathcal{N}(0; D_{ij}), \tag{4.3}$$

to the Gaussian random variables with zero mean and variance D_{ij} *and for* λ*-a.a.* $(d\lambda = \prod_{j\in\mathbf{Z}} d\nu_j)$ *configurations* $\mathbf{h}$ *each of the following event:*

$$\{\Delta_\Lambda(i,j;\mathbf{h}) > C\}\, and\, \{\Delta_\Lambda(i,j;\mathbf{h}) < -C\},$$

occurs for infinitely many terms of the sequence $\{\Delta_\Lambda(i,j;\mathbf{h})\}_\Lambda$ *for any* $C > 0$.

Proof. Expanding the function $G_\Lambda(y;\mathbf{h})$ (3.3) around the point y_{0i}, one gets

$$G_\Lambda\left(y_{0i}^{(\Lambda)}; \mathbf{h}\right) = \sum_{n=0}^{2} \frac{d^n G_\Lambda(y_{0i};\mathbf{h})}{dy^n} \frac{\left(y_{0i}^{(\Lambda)} - y_{0i}\right)^n}{n!} + O\left(y_{0i}^{(\Lambda)} - y_{0i}\right)^3. \tag{4.4}$$

Now, applying the Law of the Iterated Logarithm (see, e.g. [31]) to the sum of the i.i.d.r.v.

$$\sigma_{\Lambda,i}(\mathbf{h}(\cdot)) \equiv \frac{dG_\Lambda(y_{0i};\mathbf{h}(\cdot))}{dy} = \frac{1}{N}\sum_{l\in\Lambda}\left[y_{0i} - \tanh\beta\left(y_{0i} + h_l(\cdot)\right)\right], \tag{4.5}$$

(here $\mathbf{E}\left[\tanh\beta(y_{0i} + h_l(\cdot))\right] = y_{0i}$ by definition of y_{0i}) we obtain that $\sigma_{\Lambda,i}(\mathbf{h}) = o\left(N^{-\frac{1}{2}+\varepsilon}\right)$ for λ-a.a. configurations $\mathbf{h}$ and $\varepsilon > 0$. According to Lemma B.4, for

λ-a.a. configurations $\mathbf{h}$ we have the estimate $y_{0i}^{(\Lambda)} - y_{0i} = o\left(N^{-\frac{1}{2}+\varepsilon}\right)$. Therefore, we can represent (4.2) as follows

$$\Delta_\Lambda(i,j;\mathbf{h}(\cdot)) = \frac{1}{\sqrt{N}} \sum_{l\in\Lambda} \left[\ln \frac{\cosh\beta\,(y_{0j}+h_l(\cdot))}{\cosh\beta\,(y_{0i}+h_l(\cdot))} - \mathbf{E}\ln \frac{\cosh\beta\,(y_{0j}+h_l(\cdot))}{\cosh\beta\,(y_{0i}+h_l(\cdot))} \right] + o\left(N^{-\frac{1}{2}+\varepsilon}\right). \tag{4.6}$$

Hence, the assertions of lemma are a consequence of the Central Limit Theorem and the Law of the Iterated Logarithm. ■

Proof (of Theorem 4.1). First of all, let us note that existence of the sequence $\left\{y_{0i}^{(\Lambda)}\right\}_\Lambda$, which we need for validity of Lemma 4.1, is the assertion of Corollary B.1. Using the explicit formulas (3.2) and (3.3), we get for the "conditional" measure $\mu_\Lambda(\cdot;\mathbf{h}(\cdot))|D(\overline{\mathbf{h}}_I)$ the following representation

$$\mu_\Lambda(dy;\mathbf{h}(\cdot)|\overline{\mathbf{h}}_I) = \frac{\prod_{j\in I}\cosh\beta(y+\overline{h}_j)\,\exp\left[-\beta N G_{\Lambda\setminus I}(y;\mathbf{h}(\cdot))\right]}{\int_{\mathbf{R}^1} dy\,\prod_{j\in I}\cosh\beta(y+\overline{h}_j)\,\exp\left[-\beta N G_{\Lambda\setminus I}(y;\mathbf{h})(\cdot)\right]}dy, \tag{4.7}$$

where $G_{\Lambda\setminus I}(y;\mathbf{h}(\cdot)) = \frac{1}{2}y^2 - \frac{1}{\beta N}\sum_{j\in\Lambda\setminus I}\ln\cosh\beta(h_j(\cdot)+y)$. Then, by the Laplace method (see Lemma B.3) and (4.2) we obtain for the left-hand side of (3.15) that

$$\int_{\mathbf{R}^1} \mu_\Lambda(dy;\mathbf{h}(\cdot)|\overline{\mathbf{h}}_I)\varphi(y) = t_\Lambda(\mathbf{h}(\cdot))\varphi\left(y_{01}^{(\Lambda)}\right) + (1-t_\Lambda(\mathbf{h}(\cdot)))\,\varphi\left(y_{02}^{(\Lambda)}\right) + O\left(N^{-1}\right), \tag{4.8}$$

as $N\to\infty$, where the random variable

$$t_\Lambda(\mathbf{h}(\cdot)) = \left\{1 + \left[\frac{\partial_y^2 G_\Lambda\left(y_{01}^{(\Lambda)};\mathbf{h}(\cdot)\right)}{\partial_y^2 G_\Lambda\left(y_{02}^{(\Lambda)};\mathbf{h}(\cdot)\right)}\right]^{\frac{1}{2}} \times\right.$$

$$\left.\times \prod_{j\in I} \frac{\cosh\beta\left(y_{02}^{(\Lambda)}+\overline{h}_j\right)\cosh\beta\left(y_{01}^{(\Lambda)}+h_j(\cdot)\right)}{\cosh\beta\left(y_{01}^{(\Lambda)}+\overline{h}_j\right)\cosh\beta\left(y_{02}^{(\Lambda)}+h_j(\cdot)\right)}\exp\left[-\sqrt{N}\Delta_\Lambda(2,1;\mathbf{h}(\cdot))\right]\right\}^{-1}. \tag{4.9}$$

Therefore, using the conditions $\partial_y^2 G(y_{0i}) > 0,\ i=1,2$, Lemma B.1 and Lemma 4.1, for any $\varepsilon>0$ we can estimate by (4.9) the probability p_ε:

$$p_\varepsilon \equiv \lim_{N\to\infty} \Pr\{t_\Lambda(\mathbf{h}(\cdot)) \in [\varepsilon, 1-\varepsilon]\}$$

$$\le \lim_{N\to\infty} \Pr\{|\Delta_\Lambda(2,1;\mathbf{h}(\cdot))| < \delta\} = \frac{1}{\sqrt{2\pi D_{21}}}\int_{[-\delta,\delta]} dx\,\exp\left(-\frac{x^2}{2D_{21}}\right),$$

for any arbitrary small $\delta>0$. Hence, $p_\varepsilon = 0$ and $t_\Lambda(\mathbf{h}(\cdot)) \xrightarrow{d} t(\cdot) \in \{0;1\}$, as $\Lambda\uparrow\mathbf{Z}$, where, by symmetry of the distribution of the random variable (4.3), one gets $\Pr\{t(\cdot)=1\} = \Pr\{t(\cdot)=0\} = \frac{1}{2}$. ■

Corollary 4.1 *The random infinite-volume Gibbs state (3.14) in this case has the quasi-free form (see (3.16)), (4.1)):*

$$P_\chi(\cdot;\mathbf{h}) = t(\chi)P^{(0)}(\cdot;\mathbf{h}+y_{01}) + (1-t(\chi))P^{(0)}(\cdot;\mathbf{h}+y_{02}), \tag{4.10}$$

where $t(\cdot) = 0, 1$ *is dichotomous random variable with equal probabilities for* 0 *and* 1.

Remark 4.1. Distribution (4.10) is independent of the limiting convexity (*strength* of minima, see Definition B.1) $\lim_{\Lambda\uparrow\mathbf{Z}} \partial_y^2 G_\Lambda \left(y_{0i}^{(\Lambda)}; \mathbf{h}\right) = c_i > 0,\ i = 1, 2$, in the vicinity of the minima $\{y_{0i}\}_{i=1,2}$, as well as, of theirs positions and finite configurations $\overline{\mathbf{h}}_I, \mathbf{h}_I$, see (4.9). A more complicated situation corresponds to the case when the *type* of at least one of the minima is greater then 1, see Definition B.1.
Remark 4.2. If $\mathrm{card}\mathcal{M}(\nu, \beta) = m > 2$, then even in the case of minima of the first order we encounter a more complicated situation. Using the same arguments as above, we obtain that in the limit $\Lambda \uparrow \mathbf{Z}$

$$\mu_\Lambda(dy; \mathbf{h}(\cdot)|\overline{\mathbf{h}}_I) \overset{d}{\Longrightarrow} \sum_{i=1}^{m} t^{(i)}(\cdot)\delta(y - y_{0i})\, dy, \quad \sum_{i=1}^{m} t^{(i)}(\cdot) = 1. \tag{4.11}$$

For the coefficients $t^{(i)}(\cdot) \in \{0; 1\}$ the formula similar to (4.9) holds true. But in this case ($m > 2$), we have more than one random coefficient, see (4.11). Therefore, to get an explicit formula for $\mu(dy; \cdot)$ we have to calculate a joint distribution for these coefficients. It turns out that $\{t^{(i)}(\cdot)\}_{i=1}^m$ are dependent random variables and the calculation of the corresponding joint distribution is rather difficult problem. So, in the case $m > 2$ we get an incomplete answer on the question. The random IVGS has the form

$$P_\chi(\cdot; \mathbf{h}(\cdot)) = \sum_{i=1}^{m} t^{(i)}(\chi) P^{(0)}(\cdot; \mathbf{h}(\cdot) + y_{0i}), \quad \sum_{i=1}^{m} t^{(i)}(\chi) = 1. \tag{4.12}$$

where $t^{(i)}(\cdot) \in \{0; 1\}$.
Remark 4.3. If $\mathrm{card}\mathcal{M}(\nu; \beta) = 1$ we return to Remark 3.1 and get a (unique) random free Gibbs state (3.8).

Summarizing all we got by expressions (4.10) and (4.12), one concludes that the IVGS constructed above for the Curie-Weiss RFIM is a (quasi-free) *random mixture* of the *free states* $\left\{P^{(0)}(\cdot; \mathbf{h}(\cdot) + y_{0i})\right\}_{i=1}^{m}$, which are *pure product-measures*, see Section 2. The problem of decomposition of the IVGS into pure (extreme) Gibbs states is one of the main question of this theory [1]–[5]. The standard approach for scanning all limiting Gibbs states is to change boundary conditions $\overline{\mathbf{s}}$, see [1]–[7]. We indicated there (see also Section 2) that for the CW ferromagnet the only admissible one is the empty boundary condition which formally corresponds to $\{s_i = 0\}_{i\in\overline{\Lambda}}$. As it was discovered in [8, 9], see also [29], selection of the different IVGS of the Curie-Weiss-Ising model can be realized by the different choice of the *infinitesimal external fields* (the *generalized quasi-average method*). This means that instead of (2.3) one has to consider the perturbed Hamiltonian

$$H_\Lambda^{(\rho)} = H_\Lambda(\mathbf{s}_\Lambda; \mathbf{h}(\cdot)) + \frac{h_0}{N^\rho} \sum_{i\in\Lambda} s_i, \quad \rho > 0. \tag{4.13}$$

Then function (3.3) takes the form $G_\Lambda^{(\rho)}(y;\mathbf{h}(\cdot)) = G_\Lambda(y;\mathbf{h}(\cdot) - h_0 N^{-\rho})$. Hence, expanding $G_\Lambda(y;\mathbf{h}(\cdot) - h_0 N^{-\rho})$ one gets

$$G_\Lambda^{(\rho)}(y;\mathbf{h}(\cdot)) = G_\Lambda(y;\mathbf{h}(\cdot)) + h_0 N^{-\rho}\frac{1}{N}\sum_{j\in\Lambda}\tanh\left[\beta\left(y + h_j + \theta h_0 N^{-\rho}\right)\right], \quad 0<\theta<1. \tag{4.14}$$

By the SSLN, the last term in (4.14) does not influence on the λ-a.s. convergence of (4.14) to $G(y)$, see (3.6). But it can drastically change the limiting distribution of the variable

$$\Delta^{(\rho)}(i,j;\cdot) = d\text{-}\lim_{\Lambda\uparrow\mathbf{Z}}\Delta_\Lambda^{(\rho)}(i,j;\mathbf{h}(\cdot)) = d\text{-}\lim_{\Lambda\uparrow\mathbf{Z}}\left\{\beta\sqrt{N}\left[G_\Lambda^{(\rho)}(y_{0i}^{(\Lambda)};\mathbf{h}(\cdot)) - G_\Lambda^{(\rho)}(y_{0j}^{(\Lambda)};\mathbf{h}(\cdot))\right]\right\}. \tag{4.15}$$

Using the line of reasoning of Lemma 4.1 we distinguish the following cases:

(a) $\rho > \frac{1}{2}$. Then the external field is switching out *too fast* to change the distribution of the random variable (4.15). So, $\Delta^{(\rho)}(i,j;(\cdot)) = \mathcal{N}(0;D_{ij})$, cf. (4.3), and we get the same results (4.1) and (4.10) as for $h_0 = 0$.

(b) $\rho = \frac{1}{2}$. Then by (4.14) and Lemma 4.1 we get for (4.15) that

$$\lim_{\Lambda\uparrow\mathbf{Z}}\Delta_\Lambda^{(\rho=\frac{1}{2})}(i,j;\mathbf{h}(\cdot)) \stackrel{d}{=} \mathcal{N}(0;D_{ij}) + h_0\left(y_{0j} - y_{0i}\right). \tag{4.16}$$

Therefore, by (4.9) we obtain for the distribution of the dichotomous variable $t(\cdot)\in\{0;1\}$ the following:

$$\Pr\{t(\cdot)=1\} = \frac{1}{\sqrt{\pi D_{21}}}\int_0^\infty dx\,\exp\left\{-\frac{\left[x + h_0(y_{02}-y_{01})\right]^2}{D_{21}}\right\}; \tag{4.17}$$

$$\Pr\{t(\cdot)=0\} = 1 - \Pr\{t(\cdot)=1\}.$$

(c) $0<\rho<\frac{1}{2}$. Now the external field in (4.13),(4.14) is switching out *too slowly* and formally correspond to $h_0\to\infty$ (as $h_0 \sim N^{\frac{1}{2}-\rho}$) in the case (b). Then by (4.16) and (4.17) we get that $\left\{t_\Lambda^{(\rho)}(\mathbf{h}(\cdot))\right\}_\Lambda$ converge to a *degenerate* random variable:

$$\lim_{\Lambda\uparrow\mathbf{Z}} t_\Lambda^{(\rho)}(\mathbf{h}(\cdot)) \stackrel{\lambda\text{-a.s.}}{=} \begin{cases} 1 & \text{if } \mathrm{sign}[h_0(y_{02}-y_{01})] = 1 \\ 0 & \text{if } \mathrm{sign}[h_0(y_{02}-y_{01})] = -1 \end{cases}. \tag{4.18}$$

Consequently, the quasi-average method (4.13) gives the following limiting Gibbs states:

(a) $P_\chi^{(\rho>\frac{1}{2})}(\cdot;\mathbf{h}(\cdot)) = P_\chi(\cdot;\mathbf{h}(\cdot))$, see (4.10).

(b) $P_\chi^{(\rho=\frac{1}{2})}(\cdot;\mathbf{h}(\cdot)) = t(\chi)P^{(0)}(\cdot;\mathbf{h}(\cdot)+y_{01}) + (1-t(\chi))P^{(0)}(\cdot;\mathbf{h}(\cdot)+y_{02})$, where by the amplitude h_0 in (4.13) we can vary the distribution of the dichotomous random variable $t(\cdot)\in\{0;1\}$ from $\Pr(t(\cdot)=0)=\frac{1}{2}$ for $h_0=0$ to $\Pr(t(\cdot)=0)=0$ or 1 for $h_0(y_{02}-y_{01})\to\pm\infty$, respectively (see case (c)).

(c) $P_\chi^{(\rho<\frac{1}{2})}(\cdot;\mathbf{h}(\cdot))$ coincides, in this case, with one of the pure states $P^{(0)}(\cdot;\mathbf{h}(\cdot)+y_{01})$ or $P^{(0)}(\cdot;\mathbf{h}(\cdot)+y_{02})$ according the rule (4.18).

Thus, summarizing we get the following statement about the structure of the random IVGS for the Curie-Weiss RFIM in the case when card$\mathcal{M}(\nu;\beta)\leq 2$.

Theorem 4.2 *Assume that conditions of Theorem 4.1 are satisfied. Then by the quasi-average method (4.13) one gets that the Infinite-Volume Gibbs States for the Curie-Weiss RFIM are the random mixtures of pure (free) states:*

$$P_\chi(A;\mathbf{h}(\cdot)) = t(\chi)P^{(0)}(A;\mathbf{h}(\cdot)+y_{01}) + (1-t(\chi))P^{(0)}(A;\mathbf{h}(\cdot)+y_{02}), \quad A\in\mathcal{B}(\mathcal{S}). \tag{4.19}$$

Here the dichotomous random variable $t(\cdot)\in\{0;1\}$ has distribution (4.17) defined by the "fading out" of the infinitesimal external field in (4.13). The extreme cases $\rho>\frac{1}{2}$ and $0<\rho<\frac{1}{2}$ correspond formally to $h_0=0$ and $h_0\to\pm\infty$ respectively.

Remark 4.4. By $t(\cdot)(1-t(\cdot))=0$ one gets that for each realization of $\chi\in X$ the state (4.19) (as well as (4.12)) is *pure*. So, in contrast to the *non-random* CW model [8]–[10], the coefficients in decompositions (4.12),(4.19) never be *really mixing*, i.e., $0<t(\chi)<1$.

5. Example and Discussion

First we illustrate our results for the simple case of the dichotomous fields, with probability density

$$\frac{d\nu(h)}{dh} = 1/2\,[\delta(h-H)+\delta(h+H)]\,.$$

It is clear that in this case function $G_\Lambda(y;\mathbf{h})$, see (3.3), may be rewritten as

$$G_\Lambda(y;\mathbf{h}) = G(y) - \frac{1}{2\beta H}\left(\frac{1}{N}\sum_{i=1}^{N} h_i\right) g(y)\,, \tag{5.1}$$

where G(y) is an even function given by (3.6), and g(y) is the odd function

$$g(y) = \ln\left\{\frac{\cosh[\beta(y+H)]}{\cosh[\beta(y-H)]}\right\}.$$

Here $\beta^{-1}=\theta$ is the temperature of the system.

This model has been carefully studied by Salinas and Wreszinski [26], and it has been shown that there is $H_C=1/2$ such that for $H>H_C$ the function $G(y)$ will have only one global (quadratic) minimum of the type 1 at $y=0$, i.e., $m=1, k=1$, see Definition B.1. If the structure of the minima being so, we say that the system is in the *paramagnetic phase*.

For $0\leq H\leq H_C$ there is a decreasing continuous function $\theta_c(H)$ of critical temperatures such that $\theta_c(0)=1$ and $\theta_c(H_C)=0$; for $\theta>\theta_c(H)$ the system is in paramagnetic phase, while for $\theta < \theta_c(H)$ the function $G(y)$ has two different symmetric global minima of the type 1 ($m = 2, k_1 = k_2 = 1$). See Fig. 1.

At the curve $\theta_c = \theta_c(H)$ the situation is the following. There is $H_t > 0, H_t < H_C$, such that if $H < H_t$, $G(y)$ has one global minimum of the type 2 ($m = 1, k = 2$) at $y = 0$. If $H = H_t$, again there will be only one global minimum at $y = 0$ but of the type $k = 3$. This is a so-called *tricritical point*. Finally, if $H_t < H \leq H_C$, then $G(y)$ will have three quadratic global minima ($m = 3, k_1 = k_2 = k_3 = 1$), one at y=0 and the other two symmetric. See Fig. 1.

We now turn to the structure of the IVGS for this model first in the sense of Definition 2.2, see also Fig. 1 and 2.

In all the cases when $m = 1$ ($k = 1, 2$) we get a (unique) random Gibbs State which coincides with the free Gibbs measure $P^{(0)}(\cdot; \mathbf{h})$ because $y_0 = 0$, see Remark 4.3. The same follows immediately from the representation (5.1): $f(y_0 = 0) = 0$ and $G_\Lambda(y; \mathbf{h}(\cdot))$ is independent of $\mathbf{h}$. Hence, Definitions 2.1 and 2.2 in this case coincide.

If $m = 2$, $k_{1,2} = 1$, then the problem of the structure of the IVGS is completely resolved by the Theorem 4.1 and Corollary 4.1 (see also Remark 4.1). Again, one could deduce these results directly from the representation (5.1) and the simple structure of the random variable $\Delta_\Lambda(2, 1; \mathbf{h}(\cdot)) = \dfrac{g(y_{01})}{H} \dfrac{1}{\sqrt{N}} \sum_{j\in\Lambda} h_j(\cdot)$, $y_{02} = -y_{01} = y^* > 0$, see (4.6),(4.9) or (B.8),(B.9). Then by (5.1) the events

$$\left\{\mathbf{h} : G_\Lambda(y_{01}^{(\Lambda)}; \mathbf{h}) > (<) G_\Lambda(y_{02}^{(\Lambda)}; \mathbf{h})\right\} = \left\{\mathbf{h} : c_\Lambda = \operatorname{sign}\left(\frac{1}{\sqrt{N}} \sum_{j\in\Lambda} h_j\right) = +(-)1\right\}.$$

Therefore, by the Central Limit Theorem for $\frac{1}{\sqrt{N}} \sum_{j\in\Lambda} h_j(\cdot)$ one obtains from (4.9) that $\Pr\{t(\cdot) = 0, 1\} = 1/2$. Using the *quasi-average method* (4.13), we can change this distribution as explained in (4.17).

For $m = 3$ ($k_{1,2,3} = 1$), instead of the general theory (see Remark 4.2), it is easier to exploit the representation (5.1). Hence, $G_\Lambda(y_{02} = 0; \mathbf{h}(\cdot)) = G(y_{02})$ and this minimum has to be taken into account together with $-y_{01} = y_{03} = y^* > 0$ if the following event $\left\{\mathbf{h} : \dfrac{1}{\sqrt{N}} \sum_{j\in\Lambda} h_j \to 0\right\}$ occurs as $\Lambda \uparrow \mathbf{Z}$, see Corollary B.2. But the probability of this event is zero and by the above arguments we get

$$\lim_{\Lambda\uparrow\mathbf{Z}} \Pr\left\{\mathbf{h} : G_\Lambda(y_{01}^{(\Lambda)}; \mathbf{h}) < \min\left[G_\Lambda(y_{02}^{(\Lambda)}; \mathbf{h}), G_\Lambda(y_{03}^{(\Lambda)}; \mathbf{h})\right]\right\} =$$

$$\lim_{\Lambda\uparrow\mathbf{Z}} \Pr\left\{\mathbf{h} : G_\Lambda(y_{03}^{(\Lambda)}; \mathbf{h}) < \min\left[G_\Lambda(y_{01}^{(\Lambda)}; \mathbf{h}), G_\Lambda(y_{02}^{(\Lambda)}; \mathbf{h})\right]\right\} = \frac{1}{2}.$$

Therefore, in decomposition of the random IVGS (4.12) we get three dichotomous random variables $\left\{t^{(i)}(\cdot) = 0, 1\right\}_{i=1}^{3}$, but in contrast to Remark 4.2 the correlations between them are trivial because of $\Pr\{t^{(2)}(\cdot) = 0\} = 1$. Hence, $t^{(1)}(\cdot) = (1 - t^{(3)}(\cdot))$ and $\Pr\{t^{(1)}(\cdot) = 0, 1\} = 1/2$. Again, using the *quasi-average method* (4.13), one can change these distributions.

For this exactly soluble model we can investigate the relation between IVGS in the sense of Definition 2.2 and the *compactness arguments*, see Corollary 3.1 and Remarks 3.1,3.2 which motivate our approach to the random IVGS for Curie-Weiss RFIM.

For $m = 1$ ($k = 1, 2$) these arguments lead to the same result implied by Definition 2.1 and 2.2, see Remark 3.1. If $m = 2$ ($k_{1,2} = 1$), then by Corollary B.2 we get

$$\int_{\mathbf{R}^1} \mu_\Lambda(dy; \mathbf{h})\varphi(y) = t_\Lambda(\mathbf{h})\varphi(y_{01}^{(\Lambda)}) + (1 - t_\Lambda(\mathbf{h}))\,\varphi(y_{02}^{(\Lambda)}) + O(N^{-1}) \tag{5.2}$$

where $\Lambda \uparrow \mathbf{Z}$ and

$$t_\Lambda(\mathbf{h}) = \left\{ 1 + \left[\frac{\partial_y^2 G_\Lambda(y_{01}^{(\Lambda)}; \mathbf{h})}{\partial_y^2 G_\Lambda(y_{02}^{(\Lambda)}; \mathbf{h})} \right]^{\frac{1}{2}} \exp\left[-\sqrt{N}\Delta_\Lambda(2, 1; \mathbf{h}) \right] \right\}^{-1}. \tag{5.3}$$

For λ-a.a. fixed $\mathbf{h} \in \{-H, H\}^{\mathbf{Z}}$ and any *fixed* $k \in \mathbf{Z}^1$ one can choose a subsequence $\left\{\Lambda_n^{(k)}(\mathbf{h})\right\}_n$ so that

$$\sum_{j \in \Lambda_n^{(k)}(\mathbf{h})} h_j = kH, \quad |\Lambda_n^{(k)}(\mathbf{h})| = N_n^{(k)}(\mathbf{h}), \tag{5.4}$$

and $N_n^{(k)}(\mathbf{h}) \to \infty$, as $n \to \infty$. Below we put $\Lambda_n^{(k)}(\mathbf{h}) = \Lambda_n$ and $N_n^{(k)}(\mathbf{h}) = N_n$ for short. Then we get (see (5.1))

$$G_{\Lambda_n}(y; \mathbf{h}) = G(y) - \frac{k}{2\beta N_n} g(y), \tag{5.5}$$

and equation

$$y = \frac{1}{2}\left(1 + \frac{k}{N_n}\right)\tanh\beta(y + H) + \frac{1}{2}\left(1 - \frac{k}{N_n}\right)\tanh\beta(y - H) \tag{5.6}$$

which defines the minima $\left\{y_{01}^{(\Lambda_n)}\right\}_{i=1,2}$. Using the same line of reasoning as in the proof of Lemma B.4, we get the following results

$$y_{0i}^{\Lambda_n} = y_{0i} + \frac{1}{N_n}\alpha_i^{(n)}, \quad i = 1, 2 \tag{5.7}$$

where $\left\{\alpha_i^{(n)}\right\}$ are some bounded sequences. As it has been already mentioned above, in our case: $-y_{01} = y_{02} = y^* > 0$. Here y^* is a positive solution of the equation

$$y = \frac{1}{2}\left[\tanh\beta(y + H) + \tanh\beta(y - H)\right].$$

Therefore, when $\Lambda_n \uparrow \mathbf{Z}$, we obtain from (4.2),(5.3) and (5.7) that

$$t_{\Lambda_n}(\mathbf{h}) = \{1 + \exp[-kg(y^*)]\}^{-1} + O\left(N_n^{-1}\right). \tag{5.8}$$

Hence, in the case $m = 2$ the set $\{\mu_\alpha(dy;\mathbf{h})\}_\alpha$ of all *weak accumulation points* for the measures $\{\mu_\Lambda(dy;\mathbf{h})\}_\Lambda$ can be described as

$$\left\{(1+\exp[-kg(y^*)])^{-1}\,\delta(y+y^*) + (1+\exp[kg(y^*)])^{-1}\,\delta(y-y^*)\right\}_{k\in\mathbf{Z}^1}. \tag{5.9}$$

The corresponding family of the infinite-volume (quasi-free) Gibbs states has the form (3.7).

In the case of $m = 3$ ($k_{1,2,3} = 1$) we have $\{y_{01} = -y^*; y_{02}; y_{03} = y^*\}$. By the same choice of subsequence $\{\Lambda_n^{(k)}(\mathbf{h})\}$, (5.4), using (5.5) and the arguments of Remark 4.2 and Lemma B.3, we obtain for the family $\{\mu_\alpha(dy;\mathbf{h})\}_\alpha$ of the *weak accumulation points* in this case the following representation

$$\left\{\frac{1}{Z_k(y^*)}\left[\exp\left(-\frac{k}{2}g(y^*)\right)\delta(y+y^*) + \delta(y) + \exp\left(\frac{k}{2}g(y^*)\right)\delta(y-y^*)\right]\right\}_{k\in\mathbf{Z}^1} \tag{5.10}$$

Here $Z_k(y^*) = 1+2\cosh\left(\frac{k}{2}g(y^*)\right)$. The corresponding family of the infinite-volume (quasi-free) Gibbs states again is described by (3.7).

Now we can explain the relation between the families of accumulation points (5.9),(5.10)and the random measures

$$\mu_{\chi,m=2}(dy) = \left[t^{(1)}(\cdot)\delta(y+y^*) + \left(1-t^{(1)}(\cdot)\right)\delta(y-y^*)\right]\,dy; \tag{5.11}$$

$$\mu_{\chi,m=3}(dy) = \left[t^{(1)}(\cdot)\delta(y+y^*) + t^{(2)}(\cdot)\delta(y) + \left(1-t^{(1)}(\cdot)-t^{(2)}(\cdot)\right)\delta(y-y^*)\right],$$

where dichotomous variables $\left\{t^{(i)}(\cdot)\right\}_{i=1,2}$ are defined above. For a fixed "typical" configuration $\mathbf{h}$ one can look over *all possible accumulation points* of the sequence $\{\mu_\Lambda(dy;\mathbf{h})\}_\Lambda$ by "tuning" in an appropriate way the subsequences $\left\{\Lambda_n^{(k)}(\mathbf{h})\right\}_n$, see (5.4). Therefore, Definition 2.1 in this case is not relevant. The transition to Definition 2.2 is implied by the *release* of configuration $\mathbf{h}$. Then for any *fixed* subsequence $\{\Lambda_\alpha\}_\alpha$ one can "measure" (using the probability distribution λ on $\left(\mathbf{R}^\mathbf{Z},\mathcal{B}(\mathbf{R}^\mathbf{Z})\right)$) how "often" the sum $\sum_{j\in\Lambda_\alpha} h_j$ be in the interval $[kH,(k+\delta)H]$ for a small $\delta > 0$. But by the Central Limit Theorem this sum is of the "order" $\sqrt{N_\alpha}$. Hence, the "typical" (in probability λ) values of k in (5.9) and (5.10) are $\pm\infty$ with $\Pr = 1/2$. This immediately reduces (5.9),(5.10) to (5.11).

Finally, we discuss the problem of the *conditional self-averaging* for the Curie-Weiss RFIM, see Definition 1.1 and Remark 1.1. For $m = 2$ ($k_{1,2} = 1$) according to the line of reasoning of the Theorem 4.1 we get for the *magnetization*

$$m_\Lambda(\mathbf{h}) = \frac{1}{N}\sum_{i\in\Lambda}\sum_{\mathbf{s}^\Lambda\in\mathcal{S}^\Lambda} s_i P_\Lambda(\mathbf{s}^\Lambda;\mathbf{h}),$$

which is a random variable on the probability space $(\mathbf{R}^\mathbf{Z};\mathcal{B}(\mathbf{R}^\mathbf{Z});\lambda)$, see (3.1) and (4.8), that

$$m_\Lambda(\mathbf{h}) = \sum_{i=1,2}\frac{1}{N}\sum_{j\in\Lambda}\tanh\beta\left(y_{0i}^{(\Lambda)}+h_j\right)t_\Lambda^{(i)}(\mathbf{h}) + o(1) \tag{5.12}$$

as $N \to \infty$. Here $t_\Lambda^{(2)} = 1 - t_\Lambda^{(1)}$, (4.8). By the same theorem one also gets that the random variables

$$t_\Lambda^{(1)}(\mathbf{h}) - I_{\Delta_\Lambda(2,1;\mathbf{h})>0} \xrightarrow{\lambda} 0, \tag{5.13}$$

as $\Lambda \uparrow \mathbf{Z}$, in probability. By (5.1) and by the definition of the random variable $c_\Lambda = \text{sign}\left(\frac{1}{\sqrt{N}}\sum_{j\in\Lambda} h_j\right)$ we obtain $I_{\Delta_\Lambda(2,1;\mathbf{h})>(<)0} = I_{c_\Lambda=+(-)1}$. Here $I_{\{\cdot\}}$ is the indicator of the event $\{\cdot\}$. So, using (5.12) and (5.13) by Definition 1.1 and Remark 1.1 we obtain for the magnetization that

$$\lim_{\Lambda\uparrow\mathbf{Z}} [m_\Lambda(\mathbf{h}) - \mathbf{E}\,(m_\Lambda(\mathbf{h})|c_\Lambda)] \xrightarrow{\lambda} 0. \tag{5.14}$$

Therefore, for Λ large enough, it is close to the random variable which is a *conditional expectation* for given algebra generated by atoms $D_\pm(\Lambda) = \{\mathbf{h} : c_\Lambda(\mathbf{h}) = \pm 1\}$. This property we call the *conditional* (or partial) *self-averaging of magnetization* [29].

For $m = 3$ $(k_{1,2,3} = 1)$, see (5.10),(5.11), by the above arguments about the structure of the IVGS we get the same limit (5.14).

For $m = 1$, we say that the magnetization is *self-averaging*, since in this case $m_\Lambda(\mathbf{h})$ converges λ-a.s. to a "non-random" value, cf. (2.4) and (5.12). For finite but large enough Λ, the probability density of $m_\Lambda(\mathbf{h})$ is concentrated in a peak around this value and converges to a degenerate distribution as described in Remarks 3.1 and 4.3 for $m = 1$, $k = 1$.

Still according to these statements, if $m > 1$ then $\lim_{\Lambda\uparrow\mathbf{Z}} m_\Lambda(\mathbf{h}(\cdot)) \stackrel{d}{=} \sum_{i=1}^{m} t^{(i)}(\cdot) y_{0i}$, in distribution. However, $t^{(i)}(\cdot)$ are random variables, though they may be in a completely different probability space than the one where the random fields $\mathbf{h}$ are defined, with probabilities that we have just calculated in this simple example. Again, for finite but sufficiently large Λ, the density of probability of $m_\Lambda(\mathbf{h})$ will be concentrated in peaks around the y_{0i} with relative weights given, e.g., by the probabilities (4.17). Therefore, the limiting magnetization is now a random variable with nondegenerate distribution, i.e., it manifests the partial (or conditional) self-averaging property. An application on the *quasi-average procedure* allows one to select any pure phase and to restore the standard self-averaging property.

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Appendix A

For the reader's convenience, here we recall briefly some basic definitions and properties of the probability measures on the space of Ising spin configurations, see, e.g. [3]–[5] and [31, 33].

Suppose that for each $i \in \mathbf{Z}$ of an arbitrary integer lattice $\mathbf{Z}$, we have a copy $\mathcal{S}_i$ of the site-configuration space $\mathcal{S}_0 = \{-1, 1\}$ (Ising spin). Then the *product space* $\mathcal{S} = \prod_{i\in\mathbf{Z}} \mathcal{S}_i$ is the space of the Ising spin configurations $\mathbf{s} = \{s_i\}_{i\in\mathbf{Z}}$ for the system on $\mathbf{Z}$. For any finite subset $\Lambda \subset \mathbf{Z}$ we define a projection $\pi_\Lambda : \mathbf{s} \to \mathbf{s}^\Lambda = \{s_i\}_{i\in\Lambda}$, $|\Lambda| < \infty$ is cardinality of Λ.

Let $I_n = \{i_1, \ldots, i_n\} \subset \mathbf{Z}$. Then $C_{I_n}(B_n) = \left\{\mathbf{s} \in \mathcal{S} : \{s_i\}_{i\in I_n} \in B_n \subset \mathcal{S}^{I_n}\right\}$ is a *cylinder set* in $\mathcal{S}$ with *base* B_n and *support* I_n. By definition, $C_{I_{n+1}}(B_n \times \mathcal{S}_{i_{n+1}}) = C_{I_n}(B_n)$ and the intersection and union of two cylinder sets are again cylinder sets. If we denote by Σ the σ-algebra of the subsets in $\mathcal{S}$ generated by all cylinder sets $\mathbf{C}(\mathcal{S})$, then $(\mathcal{S}, \Sigma)$ is a *measurable space*.

Let the set $\mathcal{S}_0 = \{-1, 1\}$ be endowed with the *discrete topology* τ_0. Then the space $\mathcal{S}$ can be topologized by the *product topology* $\tau = \prod_{i\in\mathbf{Z}} \tau_i$, i.e., by the weakest topology on $\mathcal{S}$ for which all functions $\pi_i : \mathcal{S} \to \mathcal{S}_i$, $i \in \mathbf{Z}$, are continuous. The *base* of this topology consists of the sets $\left\{\bigcap_{i\in I} \left(\pi_i^{-1}[U_i]\right) : U_i \in \tau_i\right\}_I$, where $\{I\}$ are a finite subsets of $\mathbf{Z}$. The σ-algebra $\mathcal{B}(\mathcal{S})$ generated by the open sets of the product topology τ (Borel σ-algebra of $\mathcal{S}$) coincides with Σ.

By the *Tychonoff theorem*, compactness of $\mathcal{S}_0$ in the topology τ_0 implies *compactness* of $\mathcal{S}$ in the product topology τ. On the other hand, the formula $\rho(\mathbf{s}, \mathbf{s}') = \sum_{i\in\mathbf{Z}} 2^{-\|i\|}|s_i - s_i'|$, where $\|\cdot\|$ is a Euclidean norm on $\mathbf{Z}$, defines a *metric* in the space $\mathcal{S}$. The topology defined on $\mathcal{S}$ by the metric ρ coincides with the product topology τ. Hence, the space $(\mathcal{S}, \rho)$ is a compact metric space and, as a consequence, it is complete and separable.

Let $\mathcal{C}(\mathcal{S})$ denote the (Banach) space of bounded, continuous, real-valued functions $f : \mathcal{S} \to \mathbf{R}$ with norm $\|\cdot\|_{\mathcal{S}} : \sup_{\mathbf{s}\in\mathcal{S}} |f(\mathbf{s})| = \|f\|_{\mathcal{S}} < \infty$. The set of *cylinder functions* $\mathcal{C}_{\mathbf{C}}(\mathcal{S})$ consisting of $f(\mathbf{s}) = 1$ and all finite linear combinations of characteristic functions $\sum_j \lambda_j \chi_{C_j}(\mathbf{s})$, $\{C_j \in \mathbf{C}(\mathcal{S})\}_j$, be a subalgebra of $\mathcal{C}(\mathcal{S})$ which separates points of $\mathcal{S}$. By the *Stone-Weierstrass theorem*, the subalgebra $\mathcal{C}_{\mathbf{C}}(\mathcal{S})$ is $\|\cdot\|_{\mathcal{S}}$-dense in $\mathcal{C}(\mathcal{S})$.

A natural way that probability measures arise on the compact space $\mathcal{S}$ is via the *Riesz-Markov representation theorem*: all Borel probability measures $\mathcal{M}(\mathcal{S})$are in one-to-one correspondence with *positive* linear functionals $\mathcal{L}$ on the space $\mathcal{C}(\mathcal{S})$ with norms equal to one. This means that for any $l \in \mathcal{L}$ there exists a unique probability measure P on $\mathcal{B}(\mathcal{S})$ such that $l[f] = \int_{\mathcal{S}} dP\, f \equiv \langle f \rangle_P$ for any $f \in \mathcal{C}(\mathcal{S})$.

It is clear that space of the functionals $\mathcal{L}$ belong to the unit sphere of the space $\mathcal{C}^*(\mathcal{S})$ dual to the Banach space $\mathcal{C}(\mathcal{S})$. Therefore $\mathcal{M}(\mathcal{S})$ is a convex subset of the regular Borel measures corresponding to $\mathcal{C}^*(\mathcal{S})$ via the Riesz-Markov theorem.

To define limiting Gibbs states on the measurable space $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$, one has to topologize the set $\mathcal{M}(\mathcal{S})$. The natural way is to consider topologies generated on

$\mathcal{M}(\mathcal{S})$ by the ones on the space $\mathcal{C}^*(\mathcal{S})$. The most important for probability theory is the *weak-$*$ topology* (the *vague topology*): the weakest topology on $\mathcal{C}^*(\mathcal{S})$ in which all the functions $\mathcal{F}_f : l \to l[f]$, $f \in \mathcal{C}(\mathcal{S})$, are continuous. So, $\{\mathcal{F}_f\} \subset \mathcal{C}^{**}(\mathcal{S})$. The corresponding neighborhood base at $\overline{P} \in \mathcal{M}(\mathcal{S})$ is given by sets of the form: $\mathcal{N}_{\overline{P}}(f_1, \ldots, f_k; \varepsilon) =$

$$= \left\{ P \in \mathcal{M}(\mathcal{S}) : \left| \int_{\mathcal{S}} dP\, f_i - \int_{\mathcal{S}} d\overline{P}\, f_i \right| < \varepsilon,\ f_i \in \mathcal{C}(\mathcal{S}), i = 1, 2, \ldots, k \right\}.$$

This is nothing but the well-known topology of *weak-convergence* on $\mathcal{M}(\mathcal{S})$, and we write $P_n \Rightarrow P$ or $P = \text{w-lim}P_n$ if $\lim_{n\to\infty}\langle f \rangle_{P_n} = \langle f \rangle_P$ for any $f \in \mathcal{C}(\mathcal{S})$.

The importance of the weak convergence topology on $\mathcal{M}(\mathcal{S})$ becomes clear after the following

Proposition A.1 *The set $\mathcal{M}(\mathcal{S})$ is compact with respect to the topology of weak convergence and $P_n \Rightarrow P$ in $\mathcal{M}(\mathcal{S})$ if and only if $P_n(C) \to P(C)$ for all cylinder sets $C \in \mathbf{C}(\mathcal{S})$*

Proof. Let $\{P_n\}_{n\geq 1}$ be a sequence in $\mathcal{M}(\mathcal{S})$. Since any cylinder set C is defined by its finite-dimensional base B, the set $\mathbf{C}(\mathcal{S})$ is countable. Then by the diagonal sequence trick one can find an infinite subsequence $\{P_{n'}\}_{n'\geq 1}$ such that $\lim P_{n'}(C)$ exists for each $C \in \mathbf{C}(\mathcal{S})$. Therefore, the $\lim\limits_{n'\to\infty}\langle f \rangle_{P_{n'}}$ exists for all cylinder functions $f \in \mathcal{C}_{\mathbf{C}}(\mathcal{S})$. By the Stone-Weierstrass theorem, $\mathcal{C}_{\mathbf{C}}(\mathcal{S})$ is dense in $\mathcal{C}(\mathcal{S})$. Hence, the $\lim_{n'\to\infty}\langle f \rangle_{P_{n'}} = L[f]$ exists for any $f \in \mathcal{C}(\mathcal{S})$ and $L[\cdot]$ defines a non-negative linear functional on $\mathcal{C}(\mathcal{S})$ with $L[f = 1] = 1$. Then, by the Riesz-Markov theorem, there exists a measure $P \in \mathcal{M}(\mathcal{S})$ such that $L[f] = \int_{\mathcal{S}} dP\, f$. Therefore, $P_{n'} \Rightarrow P$, i.e., the set $\mathcal{M}(\mathcal{S})$ is compact with respect to weak convergence. Now, let $C \in \mathbf{C}(\mathcal{S})$, then the function $f(\mathrm{s}) = \chi_C(\mathrm{s})$ is continuous. So, $P_n \Rightarrow P$ implies $\langle \chi_C \rangle_{P_n} \to \langle \chi_C \rangle_P$, i.e., $P_n(C) \to P(C)$. To prove the converse one can again apply the Stone-Weierstrass theorem. Then the convergence $P_n(C) \to P(C)$ for all cylinder sets and the density of the subalgebra $\mathcal{C}_{\mathbf{C}}(\mathcal{S})$ in $\mathcal{C}(\mathcal{S})$ implies $P_n \Rightarrow P$. ∎

Remark A.1. The first part of Proposition A.1 follows from the *Banach-Alaoglu theorem* about weak-$*$ compactness of the unit ball in $\mathcal{C}^*(\mathcal{S})$.

Proposition A.1 is a key to the general notion of the *infinite-volume Gibbs measure (state)* on the configuration space $\mathcal{S}$ as first introduced by Minlos [6] and Ruelle [7].

Let $\mathcal{S}_{\overline{\Lambda}}(\overline{\mathrm{s}}) \equiv \left\{ \mathrm{s} \in \mathcal{S} : s_j = \overline{s}_j, j \in \overline{\Lambda} = \mathbf{Z} \setminus \Lambda \right\}$. Then the sequence $\{P_n\}_{n\geq 1}$ is specified by the extensions $\left\{\overline{P}_{\Lambda_n,\overline{\mathrm{s}}}(\cdot)\right\}_{n\geq 1}$ on $\mathcal{B}(\mathcal{S})$ of the *finite-volume Gibbs measures* $\{P_{\Lambda_n,\overline{\mathrm{s}}}(\cdot)\}_{n\geq 1}$ for increasing sets $\Lambda_n \subset \Lambda_{n+1}$, $\Lambda_n \uparrow \mathbf{Z}$. Here $\overline{P}_{\Lambda_n,\overline{\mathrm{s}}}(A) = P_{\Lambda,\overline{\mathrm{s}}}(\pi_\Lambda(A \cap \mathcal{S}_{\overline{\Lambda}}(\overline{\mathrm{s}})))$ for $A \in \mathcal{B}(\mathcal{S})$.

The *finite-volume* measure P_Λ for the temperature β^{-1} is defined by the *Gibbs ansatz*:

$$P_{\Lambda,\overline{\mathrm{s}}}(\mathrm{s}^\Lambda) = \frac{\exp\left\{-\beta H_\Lambda\left(\mathrm{s}^\Lambda|\overline{\mathrm{s}}^{\overline{\Lambda}}\right)\right\}}{\sum_{\mathrm{s}^\Lambda\in\mathcal{S}^\Lambda}\exp\left\{-\beta H_\Lambda\left(\mathrm{s}^\Lambda|\overline{\mathrm{s}}^{\overline{\Lambda}}\right)\right\}}, \tag{A.1}$$

where $H_\Lambda\left(\mathrm{s}^\Lambda|\bar{\mathrm{s}}^{\bar\Lambda}\right)$ is the Hamiltonian of the system in the finite vessel Λ with the external (boundary) condition $\bar{\mathrm{s}}^{\bar\Lambda} = \pi_{\bar\Lambda}(\bar{\mathrm{s}})$.

Definition A.1 *We say that the probability measure P on $\mathcal{B}(\mathcal{S})$ is an* infinite-volume Gibbs state *for the system (A.1) if it belongs to the closed convex hull $\mathcal{G}_\beta$ of the set of weak accumulation points of the sequence $\left\{\overline{P}_{\Lambda,\bar{\mathrm{s}}}(\cdot)\right\}_{\Lambda\subset\mathbf{Z}}$ for $\Lambda \uparrow \mathbf{Z}$.*

Hence, the triple $(\mathcal{S}, \mathcal{B}(\mathcal{S}), P)$ is the infinite-volume Gibbs probability space.
Remark A.2. Recall that a probability measure P on $\mathcal{B}(\mathcal{S})$ is called a *DLR state* (*limiting Gibbs measure* [1]–[5]) for the system (A.1) if for each finite set $\Lambda \subset \mathbf{Z}$ and configuration $\mathrm{s}^\Lambda \in \mathcal{S}^\Lambda$ the corresponding conditional probability with respect to σ-algebra $\mathcal{B}(\mathcal{S}^{\bar\Lambda})$ satisfies the *DLR equation*:

$$P\left\{\pi_\Lambda^{-1}(\mathrm{s}^\Lambda)|\mathcal{B}(\mathcal{S}^{\bar\Lambda})\right\}(\mathrm{s}) = P_{\Lambda,\mathrm{s}}(\mathrm{s}^\Lambda), \tag{A.2}$$

P-almost sure, for $\mathrm{s}^\Lambda = \mathrm{s}|\Lambda$ and $\mathrm{s}^{\bar\Lambda} = \mathrm{s}|\bar\Lambda$, cf. (A.1). For system with "bona-fide" interactions (e.g., for the Ising model with a short-range interaction) the notions of infinite-volume Gibbs state and DLR state are *equivalent.* But for the Curie-Weiss model the right-hand side of (A.2) *has no meaning* because Hamiltonian $H_\Lambda(\mathrm{s}^\Lambda|\bar{\mathrm{s}}^{\bar\Lambda})$ *does not exists* for the configurations $\mathrm{s} \in \mathcal{S}$, see (2.3). Hence, for this model we are left only with Definition A.1. Moreover, for the Curie-Weiss RFIM the situation is even more complicated, see Sections 2 and 3.

If P is a measure on $\mathcal{B}(\mathcal{S})$, we define its projections (*marginals*) $\rho_\Gamma = P \circ \pi_\Gamma^{-1}$ on $\mathcal{B}(\mathcal{S}^\Gamma)$ by $\rho_\Gamma(A) = P(\pi_\Gamma^{-1}(A))$, $A \in \mathcal{B}(\mathcal{S}^\Gamma)$. If $\Delta \subset \Gamma$, then by definition of the cylinder sets one gets $\pi_\Delta^{-1}(B) = \pi_\Gamma^{-1}(B \times \mathcal{S}^{\Gamma\setminus\Delta})$ for $B \in \mathcal{B}(\mathcal{S}^\Delta)$. Therefore, measures ρ_Δ and ρ_Γ are related by the consistency conditions:

$$\rho_\Delta(B) = \rho_\Gamma(B \times \mathcal{S}^{\Gamma\setminus\Delta}), \quad B \in \mathcal{B}(\mathcal{S}^\Delta). \tag{A.3}$$

By the *Kolmogorov theorem* one can reverse this procedure and construct on $\mathcal{B}(\mathcal{S})$ a probability measure using the marginals satisfying (A.3).

Proposition A.2 (Kolmogorov's Theorem) *Let $\{\rho_\Gamma\}_{\Gamma\subset\mathbf{Z}}$, $|\Gamma| < \infty$, be a family of probability measures on $\left\{\mathcal{B}(\mathcal{S}^\Gamma)\right\}_{\Gamma\subset\mathbf{Z}}$ which are consistent in the sense (A.3). Then there exists a* unique probability measure *P on $\mathcal{B}(\mathcal{S})$ such that $P \circ \pi_\Gamma^{-1} = \rho_\Gamma$ for all finite $\Gamma \subset \mathbf{Z}$.*

Remark A.3. It is clear, that the process described by Proposition A.1 is not just a question of reconstructing the measure on the space $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$ from its projections on the $\left\{\mathcal{B}(\mathcal{S}^\Gamma)\right\}_{\Gamma\subset\mathbf{Z}}$. On the other hand, one can obtain consistent marginals $\{\rho_\Gamma\}_{\Gamma\subset\mathbf{Z}}$ as weak accumulation points of the probability measures $\left\{\overline{P}_{\Lambda,\bar{\mathrm{s}}}(B \times \mathcal{S}^{\Lambda\setminus\Gamma})\right\}_\Lambda$ for $\Lambda \uparrow \mathbf{Z}$ and $B \in \mathcal{B}(\mathcal{S}^\Gamma)$. If now, according the Proposition A.2, one reconstructs a probability measure on $\mathcal{B}(\mathcal{S})$ from these marginals, then by Proposition A.1 it has to coincide with one of the weak accumulation point $\left\{\text{w-lim}\overline{P}_{\Lambda,\bar{\mathrm{s}}}(\cdot)\right\}$.

Appendix B

Here we list the statements about the Laplace method which are needed in the Sections 3–5. The main difference from the standard Laplace method (see, e.g., [34]) is contained in the randomness of the function

$$G_\Lambda(y;\mathbf{h}) = \frac{1}{2}y^2 - \frac{1}{\beta N}\sum_{j\in\Lambda} \ln\cosh\left[\beta(h_j+y)\right], \tag{B.1}$$

where the random field $\mathbf{h} = \{h_j\}_{j\in\mathbf{Z}}$ is the sequence of i.i.d.r.v in the probability space $(\mathbf{R}^{\mathbf{Z}}, \mathcal{B}(\mathbf{R}^{\mathbf{Z}}), \lambda)$. Here λ is the infinite-product measure: $d\lambda = \prod_{j\in\mathbf{Z}} d\nu_j$ with identical one-dimensional marginals $\nu(x) = \Pr\{h_j \le x\}$.

Lemma B.1 *Let the probability measure ν be such that $\int_{\mathbf{R}} d\nu(x)\,|x| < \infty$. Then the function $G(y) = \frac{1}{2}y^2 - \beta^{-1}\mathbf{E}_\nu\{\ln\cosh[\beta(h_j+y)]\}$, cf. (3.6), is infinitely differentiable on $\mathbf{R}$ ($G \in C^\infty(\mathbf{R})$) and*

$$t\text{-}\lim \partial_y^k G_\Lambda(y;\mathbf{h}) \overset{\lambda\text{-a.s.}}{=} \partial_y^k G(y), \quad k \ge 0, \tag{B.2}$$

uniformly on any compact $K \subset \mathbf{R}$.

Proof. The first assertion of the lemma is the consequence of analiticity of the function $\ln\cosh[\beta(z+x)]$ in the strip $|\mathrm{Im}\, z| < \dfrac{\pi}{2\beta}$, the integral representation

$$\mathbf{E}_\nu\{\ln\cosh[\beta(h_j+y)]\} = \int_{\mathbf{R}} d\nu(x)\ \ln\cosh[\beta(x+y)]$$

and the boundedness of the first moment of the measure ν. By straightforward calculations one can show that $\left|\partial_y^k\left[G_\Lambda(y;\mathbf{h}) - \frac{1}{2}y^2\right]\right| < c_k$ for arbitrary $\mathbf{h} \in \mathbf{R}^{\mathbf{Z}}$ and $y \in \mathbf{R}$. Hence, for any $k \ge 0$ the set $\left\{\partial_y^k G_\Lambda(y;\mathbf{h})\right\}_{\Lambda,\mathbf{h}}$ is a *uniformly equicontinuous family* of functions on $\mathbf{R}$. By the Strong Law of Large Numbers for i.i.d.r.v. $\left\{\partial_y^k \ln\cosh[\beta(h_j+y)]\right\}_{j\in\mathbf{Z}}$ we obtain that

$$t\text{-}\lim \partial_y^k G_\Lambda(y;\mathbf{h}) \overset{\lambda\text{-a.s.}}{=} \partial_y^k G(y)$$

for any fixed $y \in \mathbf{R}$. Let S_y denote the subset of configurations for which (B.2) is violated. Then $\lambda(S_y) = 0$. Let $K \subset \mathbf{R}$ be compact and let $Y \subset K$ be a countable dense set. Then $\lambda\left(\cup_{y\in Y} S_y\right) = 0$. Therefore, convergence (B.2) occurs for λ-a.a. configurations $\mathbf{h}$ on Y and by continuity we obtain it on K. Now the second assertion is the consequence of uniform equicontinuity of the family $\left\{\partial_y^k G_\Lambda(y;\mathbf{h})\right\}_\Lambda$ and the Arzelà-Ascoli theorem [33]. ■

Corollary B.1 *If $y_0 \in (a,b)$, $\partial_y^2 G(y_0) > 0$ and $G(y_0) < G(y)$ for $y \in (a,b)$ and $y \ne y_0$, then there is N_0 such that, for $N > N_0$ and for λ-a.a. $\mathbf{h} \in \mathbf{R}^{\mathbf{Z}}$, there is sequence $\left\{y_0^{(\Lambda)}\right\}_\Lambda \subset (a,b)$ such that $G_\Lambda(y_0^{(\Lambda)};\mathbf{h}) < G_\Lambda(y;\mathbf{h})$ for $y \in (a,b)$ and $y \ne y_0$, and $y_0^{(\Lambda)} \to y_0$, as $\Lambda \uparrow \mathbf{Z}$, λ-a.s. .*

Definition B.1 *Let $y_0 \in \mathbf{R}$ correspond to a minimum of the function $f(y)$ and*

$$f(y) = f(y_0) + \frac{\lambda}{(2k)!}(y - y_0)^{2k} + o\left[(y - y_0)^{2k}\right] \tag{B.3}$$

as $y \to y_0$. According to [34] we call k the type *and $\lambda = \partial_y^{2k} f(y_0) > 0$ the* strength *of the minimum y_0.*

Below we list the statements which we need for the proof of the Theorem 4.1. They are nothing but an extension of the standard Laplace method to the case when the measures $\{\mu_\Lambda(dy; \mathbf{h})\}_\Lambda$ are *random.*

Lemma B.2 *Let the function $G(y)$ have on the interval (a, b) the minimum y_0 of the* type *k=1. Then there is a compact $V = [-\delta, \delta], \delta > 0$ and functions $y_\Lambda = y_\Lambda(t)$, $y = y(t)$ on this domain such that: $\overline{G}_\Lambda(y_\Lambda(t)) = G_\Lambda(y_\Lambda(t) + y_0^{(\Lambda)}; \mathbf{h}) - G_\Lambda(y_0^{(\Lambda)}; \mathbf{h}) = t^2$ and $\overline{G}(y(t)) = G(y(t) + y_0) - G(y_0) = t^2, t \in V$. In addition*

$$\partial_t y_\Lambda(t = 0) = \left[\frac{2}{\partial_y^2 G_\Lambda(y_0^{(\Lambda)}; \mathbf{h})}\right]^{\frac{1}{2}} \text{ and } \partial_t^n y_\Lambda(t) \to \partial_t^n y(t), \text{ as } \Lambda \uparrow \mathbf{Z}, \text{ uniformly on } V$$

for $n \geq 0$ and λ-a.a. $\mathbf{h} \in \mathbf{R}^{\mathbf{Z}}$.

Proof. By the Taylor formula we get

$$\overline{G}_\Lambda(y) = y^2 \int_0^1 dt\,(1 - t)\partial_y^2 G_\Lambda(yt + y_0^{(\Lambda)}; \mathbf{h}) \equiv y^2 g_\Lambda(y). \tag{B.4}$$

Then by Lemma B.1 one gets that the functions $g_\Lambda(y) \in C^\infty(\mathbf{R})$ and converge uniformly on compacts to

$$g(y) = \int_0^1 dt\,(1 - t)\partial_y^2 G(yt + y_0),$$

for λ-a.a. $\mathbf{h}$. We obtain the same representation, cf. (B.4), for $\overline{G}(y) : \overline{G}(y) = y^2 g(y)$. Hence, functions $y_\Lambda(t), y(t)$ are defined by relations

$$t^2 = y^2 g_\Lambda(y), \quad t^2 = y^2 g(y). \tag{B.5}$$

Therefore, the last assertion of the lemma is a consequence of the above mentioned properties of the functions $g_\Lambda(y)$ and $g(y)$. ∎

Lemma B.3 *Let the conditions of the Lemma B.2 be satisfied and in addition $\{\varphi_\Lambda(y)\}_\Lambda \subset C^2[a, b]$. Then the integral*

$$I_\Lambda(a, b) = \int_a^b dy\,\varphi_\Lambda(y) \exp\left[-\beta N G_\Lambda(y; \mathbf{h})\right]$$

has the following asymptotic form for $N \to \infty$ and λ-a.a. $\mathbf{h} \in \mathbf{R}^{\mathbf{Z}}$:

$$I_\Lambda(a, b) = \exp\left[-\beta N G_\Lambda(y_0^{(\Lambda)}; \mathbf{h})\right] \left\{\frac{2\pi}{\beta N \partial_y^2 G_\Lambda(y_0^{(\Lambda)}; \mathbf{h})}\right\}^{\frac{1}{2}} \left[\varphi_\Lambda(y_0^{(\Lambda)}) + O\left(N^{-1}\right)\right]. \tag{B.6}$$

Proof. Let $[-\delta,\delta]$ be as in Lemma B.2 and $\overline{I}_\Lambda \equiv I_\Lambda(t_\Lambda(-\delta), t_\Lambda(\delta))$ where $t_\Lambda(y)$ is defined by the first equation (B.5). By the change of variables: $y = y_\Lambda(t) + y_0^{(\Lambda)}$, we can rewrite the integral $\overline{I}_\Lambda$ in the form (see Lemma B.2):

$$\overline{I}_\Lambda = \exp\left[-\beta N G_\Lambda(y_0^{(\Lambda)};\mathbf{h})\right] \int_{-\delta}^{\delta} dt\, \partial_t y_\Lambda(t) e^{-\beta N t^2} \varphi_\Lambda(y_\Lambda(t) + y_0^{(\Lambda)}). \tag{B.7}$$

Now, using the expansion $r_\Lambda(t) = r_\Lambda(0) + t\partial_t r_\Lambda(0) + \frac{1}{2}t^2\partial_t^2 r_\Lambda(\theta t)$, $0 < \theta < 1$, for the function $r_\Lambda(t) = \partial_t y_\Lambda(t)\varphi_\Lambda\left(y_\Lambda(t) + y_0^{(\Lambda)}\right)$, the results of the Lemma B.2 and the change of variables: $\beta N t^2 = x$, we reduce (B.7) to the right-hand side of (B.6). By the condition $y_0 \in (a,b)$ we have

$$\min_{y\in[a,b]\setminus[y_0-\varepsilon, y_0+\varepsilon]} G_\Lambda(y;\mathbf{h}) = \alpha_\Lambda > G_\Lambda(y_0^{(\Lambda)};\mathbf{h})$$

for $\varepsilon > 0$ and by Lemma B.1 there is $\Delta > 0$ such that $\alpha_\Lambda - G_\Lambda(y_0^{(\Lambda)};\mathbf{h}) \geq \Delta$ for all large enough N and any $\mathbf{h}$. Then $I_\Lambda(a,b) - \overline{I}_\Lambda = \exp\left[-\beta N G_\Lambda(y_0^{(\Lambda)};\mathbf{h})\right] O\left(e^{-N\Delta}\right)$. ∎

Corollary B.2 *Let $\{y_{0i}\}_{i=1}^m$ be the set of global minima of $G(y)$ of the equal types $\{k(y_{0i}) = 1\}_{i=1}^m$ and $\{\varphi_\Lambda(y)\}_\Lambda \subset C^2(\mathbf{R})$. Then one gets the following asymptotic form for the integral*

$$\int_{\mathbf{R}} \mu_\Lambda(dy;\mathbf{h})\varphi_\Lambda(y) = \sum_{i=1}^m \left[\varphi_\Lambda(y_{0i}^{(\Lambda)}) + O\left(N^{-1}\right)\right] \frac{W_{\Lambda,i}(y_{0i}^{(\Lambda)})}{\sum_{j=1}^m W_{\Lambda,j}(y_{0j}^{(\Lambda)})} \tag{B.8}$$

as $N \to \infty$. Here

$$W_{\Lambda,i}(y) = \exp\left[-\beta N G_\Lambda(y;\mathbf{h})\right] \left\{\frac{2\pi}{\beta N \partial_y^2 G_\Lambda(y;\mathbf{h})}\right\}^{\frac{1}{2}}, \tag{B.9}$$

and $\mathbf{h} \in \mathbf{R}^{\mathbf{Z}}$.

Finally, the fluctuations, which appear due to dependence of $G_\Lambda(y;\mathbf{h})$ on the random field configuration $\mathbf{h} \in \mathbf{R}^{\mathbf{Z}}$, are controlled by the Lemma 4.1 and

Lemma B.4 *Let y_0 be a global minimum of $G(y)$ of the type $k = 1$ and $\left\{y_0^{(\Lambda)}\right\}_\Lambda$ be the sequence of minima of the functions $\{G_\Lambda(y;\mathbf{h})\}_\Lambda$, converging to y_0 (see Corollary B.1). Then for the λ-a.a. configurations $\mathbf{h}$ we have asymptotically*

$$y_0^{(\Lambda)} - y_0 = o\left(N^{-\frac{1}{2}+\varepsilon}\right), \tag{B.10}$$

as $N \to \infty$, for an arbitrary small $\varepsilon > 0$.

Proof. Let $\psi_\Lambda(y;\mathbf{h}) = \frac{1}{N}\sum_{j\in\Lambda}\tanh\beta(h_j + y)$. Then by the definition of the points $y_0^{(\Lambda)}$ and y_0 we have

$$y_0^{(\Lambda)} - y_0 = \left[\psi_\Lambda(y_0^{(\Lambda)};\mathbf{h}) - \psi_\Lambda(y_0;\mathbf{h})\right] + \left[\psi_\Lambda(y_0;\mathbf{h}) - y_0\right] = \tag{B.11}$$

$$= \partial_y \psi_\Lambda(y_0 + \theta_\Lambda(y_0; \mathbf{h}) \left(y_0^{(\Lambda)} - y_0\right); \mathbf{h}) \left(y_0^{(\Lambda)} - y_0\right) + [\psi_\Lambda(y_0; \mathbf{h}) - y_0] \,,$$

where $0 < \theta_\Lambda(y; \mathbf{h}) < 1$ for any Λ, y and λ-a.a. $\mathbf{h}$. By Lemma B.1 and Corollary B.1 for $k = 1$ there is N_0 and the interval $[a, b] \ni y_0$ such that for $N > N_0$ one has $y_0^{(\Lambda)} \in (a, b)$ and $\partial_y \psi_\Lambda(y; \mathbf{h}) \le \gamma < 1$ for $y \in [a, b]$ and λ-a.a. $\mathbf{h} \in \mathbf{R}^{\mathbf{Z}}$. Hence, using (B.11) we obtain for the $N \to \infty$ (and λ-a.a. $\mathbf{h}$) the following asymptotic relation:

$$y_0^{(\Lambda)} - y_0 = O\left\{ \frac{1}{1-\gamma} \frac{1}{N} \sum_{j \in \Lambda} [\tanh \beta\,(h_j + y_0) - \mathbf{E}_\nu \tanh\,(h_1 + y_0)] \right\}. \qquad \text{(B.12)}$$

Now, the assertion of the lemma is the consequence of the Law of the Iterated Logarithm applied to the arithmetical mean of the i.i.d.r.v. in the right-hand side of (B.12). ∎

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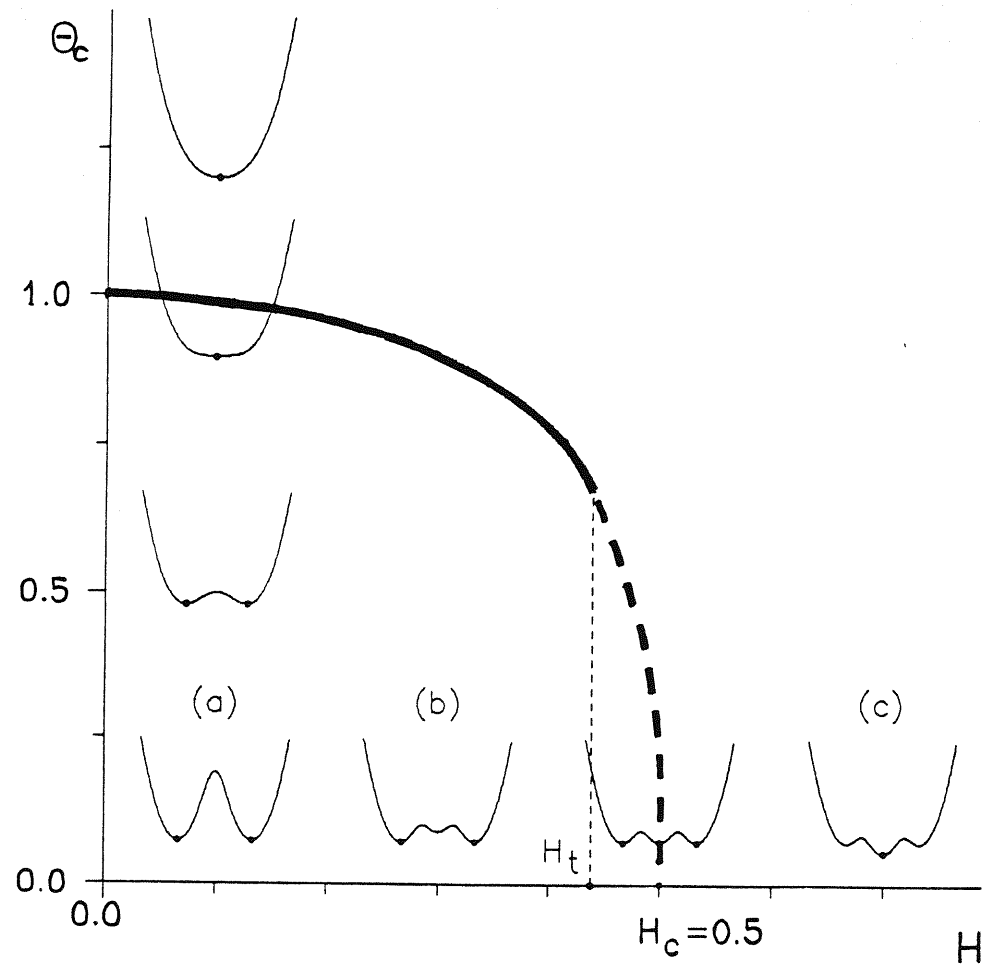

Fig 1.

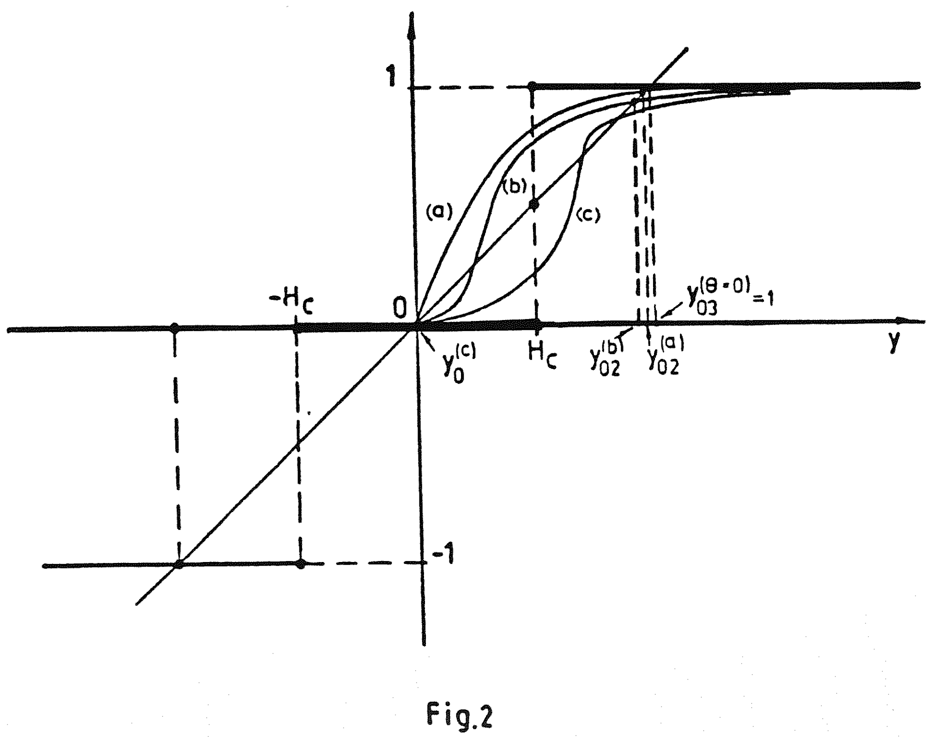

Fig.2

Figure captions

Fig.1. Phase diagram for the Curie-Weiss RFIM with $h_i = \pm H$ and the evolution of the shape of the function $G(y)$ (see (5.1)) near the bottom. The solid line corresponds to the critical points. The points on the dashed line correspond to the first-order phase transitions.

Fig.2. Solutions of the self-consistency equation

$$y^* = \frac{1}{2}\left[\tanh(\beta(y^* + H)) + \tanh(\beta(y^* - H))\right]$$

corresponding to $\partial_y G(y) = 0$, $\min_y G(y) = G(y_{0i})$. The step-function corresponds to $\beta = \infty$ and $H = H_C$. Curves (a),(b) and (c) correspond to the bottoms (a),(b) and (c) in Fig. 1 respectively. The magnetization $m(\cdot) = \sum_{i=1,2} t^{(i)}(\cdot) y_{0i}$, where $t^{(1)}(\cdot) \in \{0,1\}$ with $\Pr = \frac{1}{2}$ and $t^{(2)}(\cdot) = 1 - t^{(1)}(\cdot)$.